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Magnetohydrodynamic Shock Relations in Nonaligned Flows

Y. M. Lynn

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MAGNETOHYDRODYNAMIC SHOCK RELATIONS IN NONALIGNED FLOWS

by

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Abstract

The transition solution across a plane magnetohydrodynamic shock of finite strength in nonaligned flows in a perfect gas is investigated. A unified point of view is taken to analyze shocks in both unsteady one-dimensional flows and steady two-dimensional flows. Sets of conversion formulas connecting pertinent parameters in these two cases are given. We present a convenient parametric representation of the shock solution from which an analytical solution expressed in terms of any desirable shock strength parameter can be readily obtained by performing a straightforward calculation. Major emphasis is then placed on the analytical and numerical solutions in terms of the flow deflection angle across the shock, the parameter most often prescribed in steady two-dimensional flows; entire ranges of upstream parameters are covered. Limiting solutions for several special cases are derived. Analytical solutions in terms of the magnetic field deflection angle across the shock are also given. We illustrate the connection of the present general solution with several other major results in the literature. Dynamical stability criteria associated with general magnetohydrodynamic shocks are discussed.

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I. Introduction

The transition solution of the physical state across a plane magnetohydrodynamic (MHD) shock in a compressible, infinitely electrically conducting, inviscid, non-heat conducting perfect gas in the presence of an arbitrarily oriented uniform magnetic field is investigated. In the absence of all dissipation mechanisms, the shock appears as a surface of discontinuity with infinitesimal thickness across which the physical state undergoes a sudden jump according to conservation laws. Corresponding to each given upstream state ahead of the shock, there exists a family of downstream states behind the shock characterized by a single shock-strength parameter for which any convenient downstream physical quantity (including the shock speed or shock inclination) can be taken. Thus the transition solution consists of determining the downstream state in terms of a prescribed shock-strength parameter, together with the known upstream state. Because of the presence of Maxwell stresses, the flow direction changes across a MHD shock in general, even if it is normal to the shock surface upstream (see Section II-A); hence general MHD shocks are always oblique.

MHD shocks play an essential role in both unsteady and steady MHD flow problems when compressibility effects are duly taken into account. Their solution has been investigated by many authors. The conservation laws across a MHD shock were

were first formulated by de Hoffmann and Teller.[1] Earlier works[2-4] are mostly concerned with the classification of different types of MHD shocks, their general analytical properties and explicit transition solutions for various simple special cases. MHD shocks in aligned flows in which the magnetic field has the same direction as that of velocity on both sides of the shock were studied extensively later.[5-12] The problem is much simpler in this case; detailed basic properties, their analytical and numerical solutions, were obtained. Obviously actual MHD flows are nonaligned in general. Common justification for the study of only the restricted, and therefore simpler, case of aligned flows states that even though a general MHD flow is nonaligned, one may always transform it into an aligned flow by translating the inertia frame along the shock surface with an appropriate uniform speed to eliminate the velocity component normal to the magnetic field upstream, hence also that downstream. Strictly speaking, this is possible only for unsteady one-dimensional flows, although needless to say the inverse transformation is usually very tedious. In steady MHD flows the relative velocity between the oncoming flow and the source of disturbances (e.g., the flow velocity with respect to a fixed object downstream) is prescribed a priori and it is invariant under any transformation of inertial frame; one no longer possesses the extra degree of freedom of transforming from a nonaligned flow to an aligned flow. Besides, the shock inclination is not known and is to be determined. It is well known in steady MHD flows that if the flow is aligned at one point, it should be so throughout the entire flow. Therefore nonaligned steady flows ($\vec{B} \times \vec{U} = \vec{E} \neq 0$) and aligned steady flows ($\vec{B} \times \vec{U} = \vec{E} = 0$) represent two separate and distinct physical phenomena. The solution for MHD shocks in aligned flows actually corresponds to a very degenerate limit of the general solution for MHD shocks in nonaligned flows (see Section II-B); it cannot describe a wide variety of distinct features inherent in general MHD shocks and is by no means representative. To manifest this by a simple example, let us consider shocks of infinitesimal strength (i.e., the small disturbance waves of characteristics) in steady MHD flows. It follows immediately from the characteristic locus diagram that in aligned flows two slow shocks are always directed upstream and two fast shocks are always directed downstream, both symmetric with respect to the flow direction, whereas in general nonaligned flows, downstream-directed slow shocks and upstream-directed fast shocks exist and the symmetry property is lost.

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Solutions for MHD shocks in general nonaligned flows[13-17] have so far received only relatively little attention, mainly because of the inherent analytical complexities which arise from its involving three independent parameters characterizing any given upstream state, in addition to the ratio of specific heats (i.e., the adiabatic exponent), γ, which characterizes the gas. As a result the explicit solution and the detailed fundamental properties of general MHD shocks are only partially

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known at present. Another important point to observe is the choice of the shock strength parameter. In previous works this is mostly dictated only by analytical simplicity without due regard to the available natural parameter actually provided by the boundary condition in the specific flow problem. Although the shock solution given in terms of these analytically convenient but physically improper strength parameters do, admittedly, assume a simpler expression, the physical picture is totally obscured. In order to describe MHD shocks in actual flow problems it is essential to obtain the correct representation of the shock solution expressed in terms of an appropriate shock strength parameter in accordance with the known boundary condition. This is necessary not just because the solution may be used directly and the corresponding dynamical stability criteria can be readily examined, but also because only then the structure of the shock solution is clearly displayed, and the actual physical situation of the flow could be of help in selecting a pertinent one out of several possible solutions. Moreover, the range of the given physical parameter over which a shock solution exists can be determined conveniently from this proper representation. Due to the complexity of the solution the transformation of the analytical solution from one form to another expressed in terms of a new shock strength parameter usually presents a difficult task.

The equations for the transition solution across a plane MHD shock in unsteady one-dimensional and steady two-dimensional flows are substantially the same when they are analyzed in a coordinate system fixed to the shock. Major differences between them lie in the available upstream parameters and the shock position parameter in accordance with each of their respective physical situations, as well as the application of the solution to describe the physical flow in each case, e.g., the limited region of influence ahead of the shock on each side of the flow velocity vector which is directly associated with the direction of the latter enters into consideration only in steady two-dimensionsl flows but not in unsteady one-dimensional flows. The dynamical stability criteria for these two cases are also different. The above properties have not been emphasized previously and no explicit illustration of the connection between solutions for these two different flows has been given.

An attempt is made in this paper to remove the above mentioned inadequacies. The exact transition solution for non-relativistic, plane, compressive MHD shocks in the most general case is studied. In Section II the conservation laws across a general MHD shock are formulated and analyzed from the point of view of that in steady two-dimensional flows. The solution is given in a convenient parametric representation from which the explicit shock solution expressed in terms of any desirable shock strength parameter, as suggested by the specific flow in consideration, can be readily obtained by performing a straightforward algebraic elimination process.

Simple conversion formulas connecting pertinent parameters for shocks in steady two-dimensional flows and in unsteady one-dimensional flows are given; they serve as a link to unify the shock transition solutions for these two flows. Major emphasis is then placed on the analytical and numerical solutions in terms of the flow deflection angle across the shock which is most often prescribed either directly or indirectly from the boundary conditions in steady two-dimensional MHD flows. Limiting solutions at several special values of the basic parameters are then derived. The representation of the shock solution in terms of the magnetic field deflection angle across the shock is also given. Section III illustrates the connection of the present general solution with several other major results in the literature. The dynamical stability criteria associated with general MHD shocks are discussed in Section IV. This paper is concluded by several general discussions in Section V.

II. Formulation and Solution

The equation of state for a perfect gas (sometimes called polytropic gas) is given by $P = A(S)\rho^{\gamma}$ where γ, the ratio of specific heats (or the adiabatic exponent), is constant, P and ρ are pressure and density respectively, $A(S) = P_o/\rho_o^{\gamma} \exp(S-S_o/C_v)$ is a known function of entropy, S, with the subscript "o" denoting values at a certain given reference state, and C_v is the specific heat of gas at constant volume. The state of the gas at any point in the flow is completely described by local physical variables $(\rho, P, \vec{U}, \vec{B})$ where $\vec{U}$ and $\vec{B}$ are generally three-dimensional velocity and magnetic field vectors.

The non-relativistic limit of the well-known de Hoffmann-Teller relations across a plane stationary MHD shock (or the generalized Rankine-Hugoniot relations in magnetohydrodynamics) consists of the following system of equations which described the conservation of mass, of normal and tangential momenta, of normal magnetic field strength, of tangential electric field strength, and of energy across the shock surface.[1-4]

$$[\rho U_n] = 0 \tag{1a}$$

$$[\rho U_n{}^2 + P + \frac{B_T^2 - B_n^2}{2\mu}] = 0 \tag{1b}$$

$$[\rho U_n \vec{U}_T - \frac{B_n \vec{B}_T}{\mu}] = 0 \tag{1c}$$

$$[B_n] = 0 \tag{1d}$$

$$[U_n\vec{B}_T - \vec{U}_T B_n] = 0 \tag{1e}$$

$$[U_n\left\{\frac{\rho}{2}(U_n^2 + U_T^2) + \frac{\gamma}{\gamma - 1}P\right\} + \frac{1}{\mu}\left\{U_n B_T^2 - (\vec{U}_T \cdot \vec{B}_T)B_n\right\}] = 0 \tag{1f}$$

where [ ] denotes the jump of the quantity inside it across the shock surface, and the subscripts n and T denote normal and tangential components of the vector with respect to the shock surface. The equation of state for a polytropic gas has been used to eliminate the internal energy, e, i.e., $e = P/(\gamma-1)\rho$, in the energy equation. Since $\vec{U}_T$ and $\vec{B}_T$ are two-dimensional vectors in general, the system of (1a)-(1f) consists of eight scalar equations. Let the subscripts I and F denote the physical variables ahead of the shock and those behind the shock, respectively. $(\rho_I, P_I, \vec{U}_I, \vec{B}_I)$ are known for any given upstream state.

Define $\vec{e}_n$ to be the unit vector normal to the shock surface pointing in the direction of mass flux, so that $m = \rho_I U_{I,n} = \rho_F U_{F,n} > 0$. For general magnetic fields with $\vec{B}_I \times \vec{e}_n \neq 0$ (the special case of $\vec{B}_I \times \vec{e}_n = 0$ is discussed in IIB-i-e) we define the plane containing $\vec{e}_n$ and $\vec{B}_I$ to be the (n,t) plane and $\vec{e}_t$ to be the unit vector which makes a positive angle (counter-clockwise) $\frac{\pi}{2}$ with respect to $\vec{e}_n$ in the (n,t) plane. Also define $\vec{e}_z$ to be the unit vector normal to the (n,t) plane which, together with $\vec{e}_n$ and $\vec{e}_t$, constitutes a right-hand coordinate system, hence $\vec{e}_z = \vec{e}_n \times \vec{e}_t$. Then the shock discontinuity is in the (t,z) plane. It is evident from the definitions of $\vec{e}_t$ and

$\vec{e}_z$ that $\vec{B}_I \cdot \vec{e}_z = B_{I,z} = 0$ or $\vec{B}_{I,T} = \vec{B}_{I,t} = B_{I,t}\vec{e}_t$.

Explicit solutions of MHD shocks derived from the conservation equations, (1a)-(1f) for the special cases of contact discontinuities with $U_{I,n} = U_{F,n} = 0$ and of transverse shocks with $U^2_{F,n} = B_{F,n}/\mu\rho_F$ and $\rho_F = \rho_I$, and hence $U^2_{I,n} = U^2_{F,n} = B^2_{I,n}/\mu\rho_I$, are quite simple and well known. Only compressive shocks ($U_{I,n} \neq 0$ and excluding $U^2_{I,n} = U^2_{F,n} = B^2_{I,n}/\mu\rho_I$) which correspond to fast and slow shocks are investigated in this paper. $B_{I,n} \neq 0$ and $B_{I,t} \neq 0$ are assumed throughout the general analysis, whereas special solutions for $B_{I,n} = 0$ and $B_{I,t} = 0$ are recovered later as limits of the general solution. Since $U_{F,n} \neq \pm B_{F,n}/\sqrt{\mu\rho_I}$ and $U_{I,n} \neq \pm B_{I,n}/\sqrt{\mu\rho_I}$, it follows from (1c) and (1e) together with (1a) and (1d) that the direction of $\vec{B}_T$ always remains the same across the shock surface. Therefore $\vec{B}_F \cdot \vec{e}_z = B_{F,z} = 0$ or $\vec{B}_{F,T} = \vec{B}_{F,t}$ which means the magnetic field vectors both ahead of and behind the shock are always in the (n,t) plane. However, $\vec{U}_I$ is a three-dimensional vector and $\vec{U}_I \cdot \vec{e}_z = U_{I,z} \neq 0$ generally. Since $B_{I,z} = B_{F,z} = 0$ for a compressive magnetohydrodynamic shock, it follows from (1c) or (1e) together with (1a) and (1d) that U_z is always conserved across the shock, i.e., $U_{I,z} = U_{F,z}$. Elsewhere U_z appears only in the energy equation (1f) where, since $\rho U_n U_z^2$ remains constant across the shock, the corresponding term can be discarded. Consequently any constant value of U_z can be superimposed on both sides of the shock without affecting the transition solution of other physical variables across a compressive magnetohydrodynamic shock. Without loss of generality we consider in the following only $\vec{U} - U_z\vec{e}_z$ and $\vec{B}$. All vectors are then in the (n,t) plane and the physical phenomenon is essentially reduced to a plane one. The shock surface becomes a line of discontiniity along $\vec{e}_t$ in the (n,t) plane and it may be called the shock line simply. For later convenience we define $\vec{u} \equiv \vec{U} - U_z\vec{e}_z$, hence $u_n \equiv U_n$ and $u_t \equiv U_t$.

Apart from U_z which assumes the same constant value on both sides of the shock and can be easily taken into account, the state of the gas is now completely characterized by six scalar quantities, viz., ρ, P, u_n, u_t, B_n, B_t, in terms of which any other physical variables can be expressed. Our problem is also reduced to finding the explicit solutions of six scalar equations in the system (1a)-(1f) which govern the transition of the physical state across a plane MHD shock in both unsteady one-dimensional flows and steady two-dimensional flows. For unsteady one-dimensional flows, $\vec{e}_n$ is known since it can only be in the prescribed direction of spatial variation. The (n,t) plane is then determined by $\vec{B}_I$ and $\vec{e}_n$, $\vec{e}_t$, and $\vec{e}_z$ are obtained subsequently. For steady two-dimensional flows, one is given the ignorable direction along which the physical state does not change. Let $\vec{e}_G$ be a unit vector in this ignorable direction; the planes in which the spatial variation may take place are perpendicular to $\vec{e}_G$. Naturally $\vec{e}_n$ should be in the plane of spatial variation, hence $\vec{e}_n \perp \vec{e}_G$, which means that $\vec{e}_G$

should always be in the (t,z) plane. There are two different types of two-dimensional flows in magnetohydrodynamics. In restricted two-dimensional flows $\vec{B}_I \perp \vec{e}_G$, hence $\vec{B} \perp \vec{e}_G$ always, and it follows immediately that $\vec{e}_G \parallel \vec{e}_z$. Therefore the (n,t) plane is perpendicular to the prescribed $\vec{e}_G$ and it is known readily. The direction of $\vec{e}_t$, being the orientation of the shock line in the (n,t) plane (also giving the orientation of the shock surface (t,z) plane which is perpendicular to the (n,t) plane), is then to be determined by a single shock strength parameter and the given upstream state of the gas. In general two-dimensional flows a finite magnetic field component normal to the plane of variation is present so that $\vec{e}_G \cdot \vec{B}_I \neq 0$. Therefore $\vec{e}_G \times \vec{e}_z \neq 0$, which means that the angle between $\vec{e}_G$ and $\vec{e}_z$ in the (t,z) plane may take any value other than 0 and π. The determination of the (n,t) plane from the presecibed $\vec{e}_G$ is less direct while additional geometrical parameters should be considered.

In this paper only the shock solution in the (n,t) plane will be investigated. The result is directly applicable to describe MHD shocks in restricted two-dimensional steady flows and unsteady one-dimensional flows. The use of this solution to describe shocks in general two-dimensional flows is straightforward, since only considerations of a purely geometrical nature are needed in addition. However, the detailed analysis is rather complicated and it will not be further elaborated.

We choose to analyze the shock solution from the point of view of steady flows first. The resulting solutions are then converted to the form suitable for describing shock solutions in unsteady flows. To this end, we take the known direction of $\vec{u}_I$ as a fixed reference with respect to which all angles are measured, counterclockwise to be positive, and define β to be the shock angle made by $\vec{e}_t$, θ to be the angle made by $\vec{u}_F$ (i.e., the deflection angle of the flow direction across the shock), ψ_I and ψ_F to be the angles made by $\vec{B}_I$ and $\vec{B}_F$ respectively as shown in Fig. 1. In terms of polar coordinates with the direction of $\vec{u}_I$ being the polar axis, we have $\vec{u}_I = (u_I, 0)$, $\vec{u}_F = (u_F, \theta)$, $\vec{B}_I = (B_I, \psi_I)$, and $\vec{B}_F = (B_F, \psi_F)$ where in each bracket the first term is the magnitude of the vector and the second term is the angle made by the vector with $\vec{u}_I$. It may be seen from Fig. 1 that ahead of the shock, $u_{I,n} = u_I \sin \beta$, $u_{I,t} = u_I \cos \beta$, $B_{I,n} = B_I \sin(\beta-\psi_I)$ and $B_{I,t} = B_I \cos(\beta-\psi_I)$, while behind the shock, $u_{F,n} = u_F \sin(\beta-\theta)$, $u_{F,t} = u_F \cos(\beta-\theta)$, $B_{F,n} = B_F \sin(\beta-\psi_F)$ and $B_{F,t} = B_F \cos(\beta-\psi_F)$. The following dimensionless variables are introduced to normalize the magnitudes of the downstream physical variables with respect to those upstream, viz., $\bar{\rho} = \rho_F/\rho_I$, $\bar{P} = P_F/P_I$, $\bar{u} = u_F/u_I$ and $\bar{B} = B_F/B_I$. These, together with the angles β, θ, and ψ_F, constitute seven dimensionless parameters. Any one, or a combination of them, should be prescribed as the shock strength parameter; the remaining six independent variables describe completely the downstream state behind the shock. The given upstream state is governed by three independent dimensionless parameters defined by

$M_I = u_I/a_I$, $k_I = b_I^2/a_I^2$ and ψ_I, where $a = \sqrt{(\partial P/\partial \rho)_S} = \sqrt{\gamma P/\rho}$ is the gasdynamic sound speed and $b = B/\sqrt{\mu\rho}$ is the Alfvén wave speed. M is the Mach number and M_I measures the upstream flow speed. $k = \frac{2}{\gamma} \times$ (magnetic pressure/gas pressure), hence k_I measures the upstream magnetic field strength. ψ_I characterizes the upstream magnetic field direction. The adiabatic exponent, γ, also enters here as a parameter which depends on the nature of the gas considered and is known at the outset. It may be remarked here that the streamlines in MHD flows cannot be replaced by solid boundaries in general, for the magnetic field boundary condition should also be considered.

Expressed in terms of the dimensionless quantities defined above, (1a)-(1f) become[17] respectively,

$$\sin\beta = \bar{\rho}\bar{u}\sin(\beta - \theta) \tag{2a}$$

$$\frac{1}{\gamma} + M_I^2\sin^2\beta + \frac{k_I}{2}\cos 2(\beta - \psi_I) = \frac{\bar{P}}{\gamma} + M_I^2\bar{\rho}\bar{u}\sin^2(\beta-\theta) + \frac{k_I\bar{B}^2}{2}\cos 2(\beta-\psi_F) \tag{2b}$$

$$M_I^2\sin 2\beta - k_I\sin 2(\beta-\psi_I) = M_I^2\,\bar{\rho}\bar{u}^2\sin^2(\beta-\theta) - k_I\bar{B}^2\sin 2(\beta-\psi_F) \tag{2c}$$

$$\sin(\beta-\psi_I) = \bar{B}\sin(\beta-\psi_F) \tag{2d}$$

$$\sin\psi_I = \bar{u}\bar{B}\sin(\psi_F-\theta) \tag{2e}$$

$$\left(\frac{M_I^2}{2} + \frac{1}{\gamma-1}\right)\sin\beta + k_I\sin\psi_I\cos(\beta-\psi_I) \tag{2f}$$

$$= \bar{u}\left[\left(\frac{M_I^2\bar{\rho}\bar{u}^2}{2} + \frac{\bar{P}}{\gamma-1}\right)\sin(\beta-\theta) + k_I\bar{B}^2\sin(\psi_F-\theta)\cos(\beta-\psi_F)\right].$$

(A) Solution in terms of $\bar{\rho}$

We present the shock solution expressed in terms of the density ratio, $\bar{\rho}$, which is taken as the prescribed shock strength parameter. However, a shock in steady flows is most conveniently characterized by its orientation, e.g., the inclination angle of the shock line with respect to the given upstream flow direction, β, which determines the position of the shock line and marks geometrically the boundary in a flow field separating the given upstream state and the unknown state of gas downstream. Other downstream physical variables to be evaluated subsequently can then be determined in their respective regions in the flow. For cases where β is not prescribed it is always desirable to determine β first from the prescribed shock strength parameter together with the given upstream parameters, M_I, k_I, and ψ_I. To this end, a relation between β and $\bar{\rho}$ should be first obtained.

It is obvious, by the definitions of $\vec{e}_n$, $\vec{e}_t$, and β that $\pi \geq \beta \geq 0$ always. Instead of ψ_I, ψ_F, θ and β, it is convenient to use the tangent of these angles defined by

$$s = \tan \psi_I, \quad \phi = \tan \psi_F, \quad \varepsilon = \tan \theta, \quad \tau = \tan \beta. \qquad (3)$$

However, ϕ and ε characterize only the inclination angles (or orientations) of the unknown downstream vectors $\vec{B}_F$ and $\vec{u}_F$ with respect to $\vec{u}_I$; the senses of these vectors along their orientations which depend on the directions of $\vec{u}_I$ and $\vec{B}_I$ should also be found. With the aid of (3), the cosines of angles are given by

$$\cos\psi_I = \mathrm{sg}(\psi_I)(1+s^2)^{-\frac{1}{2}}, \quad \cos\psi_F = \mathrm{sg}(\psi_F)(1+\phi^2)^{-\frac{1}{2}},$$
$$\cos\theta = \mathrm{sg}(\theta)(1+\varepsilon^2)^{-\frac{1}{2}} \quad , \quad \cos\beta = \mathrm{sg}(\beta)(1+\tau^2)^{-\frac{1}{2}} \quad , \tag{4}$$

where $\mathrm{sg}(\xi)$ denotes the sign of $\cos\xi$ and equals +1 if $\frac{\pi}{2} > \xi > -\frac{\pi}{2}$, while it equals -1 if $\pi \geq \xi > \frac{\pi}{2}$ or $-\frac{\pi}{2} > \xi \geq -\pi$. Expressed in terms of the new variables defined by (3) and (4), (2a) and (2d) become

$$\bar{u} = \mathrm{sg}(\theta)\,\frac{\tau\sqrt{1+\varepsilon^2}}{\bar{\rho}(\tau-\varepsilon)} \tag{5a}$$

and

$$\bar{B} = \mathrm{sg}(\psi_I)\mathrm{sg}(\psi_F)\,\frac{(\tau-s)\sqrt{1+\phi^2}}{(\tau-\phi)\sqrt{1+s^2}} \quad , \tag{5b}$$

respectively. Since u and $\bar{\rho}$ are scalars, (5a) yields the condition for $\mathrm{sg}(\theta)$ that

$$\mathrm{sg}(\theta)(\tau-\varepsilon)\tau \geq 0 \tag{6a}$$

which corresponds to

$$\beta \geq \theta \geq \beta-\pi \quad . \tag{6b}$$

Similarly, since B is a scalar, (5b) yields the condition for $\mathrm{sg}(\psi_F)$ that

$$\mathrm{sg}(\psi_I)\mathrm{sg}(\psi_F)(\tau-s)(\tau-\phi) \geq 0 \tag{7a}$$

which corresponds to

$$\text{if}\quad \beta \geq \psi_I \geq \beta - \pi\ ,\ \text{then}\ \beta \geq \psi_F \geq \beta - \pi$$

$$\text{if}\quad \pi - \beta \geq \psi_I \geq \beta \quad \text{or} \quad \pi - \beta \geq \psi_I \geq \beta - 2\pi \qquad (7b)$$

$$\text{then}\quad \pi - \beta \geq \psi_F \geq \beta, \quad \text{or}\ \beta - \pi \geq \psi_F \geq \beta - 2\pi$$

with $sg(\psi_I)$ being known from the given upstream state. Therefore the senses of $\vec{u}_F$ and $\vec{B}_F$ along their orientations which are characterized by $sg(\theta)$ and $sg(\psi_F)$, respectively, are determined from (6) and (7). In fact (6) and (7) follow directly from (1a) and (1d) respectively; they describe the simple physical facts as one may observe from (1a) and (1d) readily that $\vec{u}_F$ points in the direction of the positive n-side of the shock line as $\vec{u}_I$ does while both $\vec{B}_I$ and $\vec{B}_F$ point to the same side of the shock line which can be either the positive n-side or the negative n-side. Thus $sg(\theta)$ and $sg(\psi_F)$ describing the senses of $\vec{u}_F$ and $\vec{B}_F$ respectively can be found very easily. Using (5a) and (5b) to eliminate $\vec{u}$ and $\vec{B}$ in (2b), (2c), (2e), and (2f) first and expressing the result in terms of the new variables defined in (3) and (4), we solve and get the following algebraic equations (see Appendix A):

$$f(\tau,\bar{\rho}) = k_I\bar{\rho}(\bar{\rho}-1)M_I^4\tau^4(1+s^2)(1+s\tau)^2 + [M_I^2\tau^2(1+s^2) - k_I\bar{\rho}(\tau-s)^2] \times$$
$$\Big\{2\bar{\rho}(1+\tau^2)[M_I^2\tau^2(1+s^2) - k_I\bar{\rho}(\tau-s)^2] - M_I^2\tau^2(1+s^2)[\gamma+1-(\gamma-1)\bar{\rho}] \times$$
$$[M_I^2\tau^2 - k_I\bar{\rho}(1+\tau^2)]\Big\} = 0 \qquad (8a)$$

$$\varepsilon(\tau,\bar{\rho}) = \frac{\tau(\bar{\rho}-1)[M_I^2\tau^2(1+s^2) + k_I\bar{\rho}s(1+\tau^2)(\tau-s)]}{M_I^2\tau^2(1+s^2)(\tau^2+\bar{\rho}) + k_I\bar{\rho}(1+\tau^2)(\tau-s)(\bar{\rho}s-\tau)} \tag{8b}$$

$$\phi(\tau,\bar{\rho}) = \frac{M_I^2\tau^2(1+s^2)[\bar{\rho}\tau(1+s\tau)-(\tau-s)] - k_I\bar{\rho}s(1+\tau^2)(\tau-s)^2}{M_I^2\tau^2(1+s^2)[\bar{\rho}(1+s\tau)+\tau(\tau-s)] - k_I\bar{\rho}(1+\tau^2)(\tau-s)^2} \tag{8c}$$

$$\bar{P}(\tau,\bar{\rho}) = 1 + \frac{\gamma(\bar{\rho}-1)M_I^2\tau^2}{\bar{\rho}(1-\tau^2)} \times$$

$$\left\{1 + \frac{k_I\bar{\rho}(1+s\tau)^2[2k_I\bar{\rho}(\tau-s)^2-(\bar{\rho}+1)M_I^2\tau^2(1+s^2)]}{2[M_I^2\tau^2(1+s^2) - k_I\bar{\rho}(\tau-s)^2]^2}\right\} \tag{8d}$$

$$\bar{u}^2(\tau,\bar{\rho}) = \frac{1}{1+\tau^2}\left\{\frac{\tau^2}{\bar{\rho}^2} + \left[1 + \frac{k_I\tau(\bar{\rho}-1)(\tau-s)(1+s\tau)}{M_I^2\tau^2(1+s^2)-k_I\bar{\rho}(\tau-s)^2}\right]^2\right\} \tag{8e}$$

$$\bar{B}^2(\tau,\bar{\rho}) = \frac{1}{(1+\tau^2)(1+s^2)}\left\{(\tau-s)^2 + \left\{\frac{\bar{\rho}(1+s\tau)[M_I^2\tau^2(1+s^2)-k_I(\tau-s)^2]}{M_I^2\tau^2(1+s^2)-k_I\bar{\rho}(\tau-s)^2}\right\}^2\right) \tag{8f}$$

Note that sg(ψ_I), sg(ψ_F) and sg(θ) do not appear in the above equations because only scalar physical variables and the orientations of vectors (the tangent of their angles) are involved. The complete solution for the downstream state should include sg(θ) and sg(ψ_F) determined from (6) and (7).

As far as (8a)-(8f) are concerned, it suffices to confine our attention to the consideration of the angles ψ_I, ψ_F and θ within one period, π, which will be taken here as ranging from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Equations (8a)-(8f) are invariant under the following transformation:

$$\psi_I \longrightarrow -\psi_I \ , \quad \psi_F \longrightarrow -\psi_F \ , \quad \theta \longrightarrow -\theta \ , \quad \beta \longrightarrow \pi-\beta \qquad (9a)$$

which corresponds to

$$s \longrightarrow -s \ , \quad \phi \longrightarrow -\phi \ , \quad \varepsilon \longrightarrow -\varepsilon \ , \quad \tau \longrightarrow -\tau \ . \qquad (9b)$$

Thus, without loss of generality, the upstream parameter ψ_I can be further restricted to the range from 0 to $\frac{\pi}{2}$ in the study of general shock solutions. The solution for $0 \geq \psi_I \geq -\frac{\pi}{2}$ can be easily obtained from the former by the use of the transformation specified in (9).

Equation (8a) is a polynomial of degree six in τ or a cubic equation in $\bar{\rho}$. For any prescribed value of $\bar{\rho}$, τ can be determined from (8a) which yields in general six distinct real roots of τ. Alternatively one may also solve $\bar{\rho}$ from (8a) which yields generally three distinct real roots provided τ is specified as the shock strength parameter. A selection principle based on entropy increase consideration according to the second law of thermodynamics, the dynamical stability criteria, and the relevant boundary conditions in the flow are needed to single out an appropriate solution to describe the flow in question (see Section IV).

Unlike the case in one-dimensional unsteady flows, shock lines in the (n,t) plane in steady flows to not extend indefinitely in both directions along the t-axis. In particular, at the foremost end of the extraneously introduced source of disturbances, say O, which encounters the oncoming flow first (e.g., the leading edge of a pointed body or a line current sheet) only semi-infinite attached shock lines may exist on each side of $\vec{u}_I$. Let (P) denote the positive side of $\vec{u}_I$ being the region swept by vectors making angles ranging from 0 to π with respect to $\vec{u}_I$ and (N) denote the negative side of $\vec{u}_I$ being the region swept by vectors making angles ranging from $-\pi$ to 0 with respect to $\vec{u}_I$. On the (P) side the shock line originates from 0 and extends indefinitely along the positive t-axis, whereas on the (N) side the shock line originates from 0 and extends indefinitely along the negative t-axis. Naturally the shock angles in (P) and (N) are different in general. Moreover, the shocks may be either downstream-directed if $\frac{\pi}{2} < \beta < \pi$ in (N) or $0 < \beta < \frac{\pi}{2}$ in (P), or upstream-directed if $0 < \beta < \frac{\pi}{2}$ in (N) of $\frac{\pi}{2} < \beta < \pi$ in (P). After $\bar{\rho}$ and τ are known, ε, ϕ, $\bar{P}$, $\bar{u}^2$, and $\bar{B}^2$ can be easily obtained from the auxiliary equations (8b)-(8f); they serve to determine the downstream state completely.

Note that (8a)-(8f) are all polynomials of τ; more explicit expressions of ε, ϕ, $\bar{P}$, $\bar{u}^2$, and $\bar{B}^2$ given in terms of $\bar{\rho}$ alone can be obtained by elimination of τ in (8b)-(8f) with the aid of (8a). Apart from their advantages for analytical studies, the completely explicit expressions are not necessary in actual numerical computations, since the value of τ for any given $\bar{\rho}$ can always be determined first from (8a).

In unsteady one-dimensional flows the shock is most conveniently characterized by its speed of propagation. The shock speed, together with the upstream state of the gas, serves to determine the shock completely. Thus in a coordinate system fixed to the shock front, one is given not only M_I, k_I and s, but also β (hence τ) which is the angle made by the flow velocity ahead of the shock with the stationary shock line. The latter can only be normal to the prescribed direction of spatial variation, i.e., $\vec{e}_n$. Here it is convenient to replace τ, M_I, k_I and s by the following four dimensionless parameters corresponding to ratios of various upstream velocity components with respect to the stationary shock line and the speed of sound[18]

$$\frac{u_n}{a} = \frac{M_I \tau}{(1+\tau^2)^{\frac{1}{2}}} \quad , \quad \frac{u_t}{a} = \frac{M_I}{(1+\tau^2)^{\frac{1}{2}}} \quad ,$$

$$\frac{b_n}{a} = \mathrm{sg}(\beta)\,\mathrm{sg}(\psi_I)\frac{k_I^{\frac{1}{2}}(\tau - s)}{(1+\tau^2)^{\frac{1}{2}}(1+s^2)^{\frac{1}{2}}} \quad , \quad \frac{b_t}{a} = \mathrm{sg}(\beta)\,\mathrm{sg}(\psi_I)\frac{k_I^{\frac{1}{2}}(1+\tau s)}{(1+\tau^2)^{\frac{1}{2}}(1+s^2)^{\frac{1}{2}}} \tag{10a}$$

or, conversely,

$$\tau = \frac{u_n}{u_t} \; , \; M_I^2 = \frac{u_n^2}{a^2} + \frac{u_t^2}{a^2} \; , \; k_I = \frac{b_n^2}{a^2} + \frac{b_t^2}{a^2} \; , \; s = \frac{u_n b_t - u_t b_n}{u_n b_n + u_t b_t} \tag{10b}$$

where the subscript "I" for all upstream speeds has been dropped for brevity. This abbreviated notation will be employed throughout the rest of this paper. Note that sg(β) and sg(ψ_I) do not appear in (10b). Of the four parameters given in (10a), only u_t/a, b_n/a, and b_t/a are given parameters characterizing the upstream state. They are independent of any translation of the coordinate system in the direction of $\vec{e}_n$. u_n represents the relative velocity between the shock propagation velocity and the normal component of flow velocity upstream in any coordinate system. Thus the shock speed has been incorporated into u_n. u_n/a should properly be regarded as a shock strength parameter; it plays the same role τ does in steady flows. Incidentally, in the limit of shocks of very weak strength, u_n and τ describe the characteristics in unsteady and steady flows respectively; they are basically different from all other downstream parameters in each case.

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[18] Because of the absence of any characteristic length or characteristic time, the shock solution is homogeneous in speeds and only various speed ratios enter as basic characterizing parameters.

If u_n is not prescribed in a flow problem, it is always desirable to determine u_n first from the prescribed shock strength parameter, together with the given upstream parameters, u_t/a, b_n/a, and b_t/a. The shock coordinate system is then determined and the shock propagation speed is known. Solutions for other physical variables follow readily.

Substituting (10b) into (8a)-(8f) we get the following alternative representation of the solution expressed in terms of new parameters which are suitable for describing shocks in unsteady one-dimensional flows:

$$f^*(u_n,\bar{\rho}) = \bar{\rho}(\bar{\rho}-1)u_n^4 b_t^2 + (u_n^2-\bar{\rho}b_n^2)\ 2\bar{\rho}a^2(u_n^2-\bar{\rho}b_n^2) - u_n^2[\gamma+1-(\gamma-1)\bar{\rho}](u_n^2-\bar{\rho}b^2) = 0 \qquad (11a)$$

$$\varepsilon(u_n,\bar{\rho}) = \frac{u_n(\bar{\rho}-1)[u_t(u_n^2-\bar{\rho}b_n^2) + \bar{\rho}u_n b_n b_t]}{(u_n^2+\bar{\rho}u_t^2)(u_n^2-\bar{\rho}b_n^2) + \bar{\rho}(\bar{\rho}-1)u_n u_t b_n b_t} \qquad (11b)$$

$$\phi(u_n,\bar{\rho}) = \frac{\bar{\rho} u_n b_t(u_n^2-b_n^2) - u_t b_n(u_n^2-\bar{\rho} b_n^2)}{\bar{\rho} u_t b_t(u_n^2-b_n^2) + u_n b_n(u_n^2-\bar{\rho} b_n^2)} \qquad (11c)$$

$$\bar{P}\,(u_n,\bar{\rho}) = 1 + \frac{\gamma u_n^2(\bar{\rho}-1)}{\bar{\rho} a^2}\left\{1 + \frac{\bar{\rho} b_t^2[2\bar{\rho} b_n^2 - (\bar{\rho}+1)u_n^2]}{2(u_n^2 - \bar{\rho} b_n^2)^2}\right\} \qquad (11d)$$

$$\bar{u}^2(u_n,\bar{\rho}) = \frac{1}{u^2}\left\{\frac{u_n^2}{\bar{\rho}^2} + [u_t + \frac{(\bar{\rho}-1)u_n b_n b_t}{u_n^2 - \bar{\rho} b_n^2}]^2\right\} \qquad (11e)$$

$$\bar{B}^2(u_n,\bar{\rho}) = \frac{1}{b^2}\left\{b_n^2 + [\frac{\bar{\rho} b_t(u_n^2 - b_n^2)}{u_n^2 - \bar{\rho} b_n^2}]^2\right\} \qquad (11f)$$

sg(ψ_I), sg(ψ_F) and sg(θ) do not appear in the above equations whereas their consequences can be easily accounted for as discussed before. (11a) is a cubic equation for both u_n^2 and $\bar{\rho}$. With $\bar{\rho}$ prescribed as the shock strength parameter, u_n can be determined readily from (11a) and it yields the shock propagation speed. Alternatively, if the shock propagation speed is prescribed as the shock strength parameter so that u_n is known, one may solve $\bar{\rho}$ from (11a). Other downstream variables are determined subsequently from the auxiliary equation in (11b)-(11f).

The second law of thermodynamics requires that the entropy should not decrease across a physical shock, i.e., $\bar{S} = S_F/S_I \geq 1$. Lüst[2] has shown that across a MHD shock $\bar{\rho} > 1$ if and only if $\bar{S} > 1$. Thus we may examine the entropy increase condition for a MHD shock of finite strength by considering $\bar{\rho} > 1$ simply (except for the special case of $\bar{\rho} \longrightarrow 1$ and $u_n^2 \longrightarrow b_n^2$ simultaneously, see Section IIB-i-g and Appendix D-g) which can be readily accounted for from the present representation of the shock solution.

General compressive MHD shocks can be classified into fast and slow shocks according to

$$\overline{B}_t > 1 \quad \text{and} \quad \overline{B}_t < 1 \tag{12}$$

respectively.[19] We have, by definition, $\overline{B}_t = B_{F,t}/B_{I,t} = \tan(\beta-\psi_I)/\tan(\beta-\psi_F) = (\tau-s)(1+\tau\phi)/(\tau-\phi)(1+\tau s)$ where ϕ can be eliminated by the use of (8c). Then

$$\overline{B}_t - 1 = \frac{M_I^2\tau^2(\bar{\rho}-1)(1+s^2)}{M_I^2\tau^2(1+s^2) - \bar{\rho}k_I(\tau-s)^2} = \frac{(\bar{\rho}-1)u_n^2}{u_n^2 - \bar{\rho}b_n^2} \quad . \tag{13}$$

Since $\bar{\rho} > 1$ for all MHD shocks of finite strength satisfying the entropy increase condition, (12) and (13) yield

$$u_n^2 > \bar{\rho}b_n^2 \quad \text{and} \quad u_n^2 < \bar{\rho}b_n^2 \tag{14a}$$

for fast and slow shocks in unsteady one-dimensional flows, and

$$M_I^2\tau^2(1 + s^2) > \bar{\rho}k_I(\tau-s)^2 \text{ and } M_I^2\tau^2(1+s^2) < \bar{\rho}k_I(\tau-s)^2 \tag{14b}$$

for fast and slow shocks in steady two-dimensional flows. By the use of (1a) and (1d), (14a) becomes

$$u_{F,n}^2 > b_{F,n}^2 \quad \text{and} \quad u_{F,n}^2 < b_{F,n}^2 \tag{15a}$$

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[19] This basis of classification can be refined after considering the dynamical stability criteria. The refined classification for unsteady one-dimensional MHD shocks in aligned flows with evolutionary condition taken into account has been given by Polovin in Ref. 12 (see Section IV for further discussions).

and (14b) becomes

$$M_F^2\tau^2(1+\phi^2) > k_F(\tau-\phi)^2 \quad \text{and} \quad M_F^2\tau^2(1+\phi^2) < k_F(\tau-\phi)^2 \qquad (15b)$$

In the limit of very weak shocks $\rho \rightarrow 1$, (14a and b) or (15a and b) reduce to the usual definitions for fast and slow characteristics (or small disturbance waves) given by limiting values of u_n and τ respectively.

Equations (11a), (11d), and (11f) show that $\bar{\rho}$, $\bar{P}$, and $\bar{B}^2$ do not depend on u_t. This is obvious physically, for the scalar variables (except u^2) should be invariant under any Galilean transformation of inertia frame along the shock surface (in the t-direction), whereas only ε, ϕ, and $\bar{u}^2$ could alter. In particular, if the fluid velocity ahead of the shock is perpendicular to the shock line, then $u_t = 0$. This is the case often encountered in unsteady one-dimensional flows; e.g., if one considers the shock propagating into a region otherwise at rest, the shock speed is then given by $u_n = u$. Setting $u_t = 0$ and $u_n = u$ in (11b), (11c), and (11e) yields (see also Appendix C)

$$\varepsilon = \frac{b_n b_t \bar{\rho}(\bar{\rho}-1)}{u^2-\bar{\rho}b_n^2}, \quad \phi = \frac{\bar{\rho}b_t(u^2-b_n^2)}{b_n(u^2-\bar{\rho}b_n^2)}, \quad \bar{u}^2 = \frac{1}{\bar{\rho}^2} + \left[\frac{b_n b_t(\bar{\rho}-1)}{u^2-\bar{\rho}b_n^2}\right]^2 \qquad (16)$$

Note that $\bar{u}^2 = \bar{\rho}^{-2}(1+\varepsilon^2)$ here. For MHD shocks of finite strength $\bar{\rho} > 1$ or $\bar{\rho} \rightarrow 1$ and $u_n^2 \rightarrow b_n^2$ (see Section IIB-i-g); hence even if $u_t = 0$, $\varepsilon \neq 0$ in general unless $B_n = 0$ or $B_t = 0$. Therefore, in contrast to the familiar case in gasdynamics ($B_n = B_t = 0$) in which normal shocks may exist and correspond to the case of $u_t = 0$ and then by (16) $\varepsilon = 0$, normal MHD shocks ($u_t = \varepsilon = 0$) occur only in special cases of either $B_n = 0$ or $B_t = 0$, and general MHD shocks are always oblique. This is clear physically because the presence of a magnetic field generally results in different tangential momentum and hence different tangent velocity (for the mass flux is conserved across the shock) on two sides of a MHD shock.

$\bar{\rho}$ is not the shock parameter usually prescribed in physical flow problems. Nevertheless the representation of the shock solution in terms of $\bar{\rho}$, as shown in (8a)-(8f) or (11a)-(11f), has the main advantage of having a simple analytical form from which an exact solution for the MHD shock transition expressed explicitly in terms of any other desirable shock strength parameter, say η, can be obtained readily where η can be one of τ (or u_n for unsteady one-dimensional flows), ε, $\emptyset$, $\bar{P}$, $\bar{u}^2$, and $\bar{B}^2$, or any other parameter which can always be expressed as a combination of them including $\bar{\rho}$. For steady flows, apart from the trivial case of $\eta = \tau$, the procedure is to obtain first $\eta = \eta(\tau,\bar{\rho})$ with the aid of (8a)-(8f). Using (8a) to eliminate $\bar{\rho}$ in $\eta(\tau,\bar{\rho})$ yields $f_1(\tau,\eta) = 0$. Then $f_2(\bar{\rho},\eta) = 0$ is obtained from (8a) by eliminating τ with the aid of $f_1(\tau,\eta) = 0$. All other physical variables expressed explicitly in terms of η can be determined from (8a)-(8f) with their dependence on τ and $\bar{\rho}$ being eliminated by the use of $f_1(\tau,\eta) = 0$ and $f_2(\bar{\rho},\eta) = 0$, respectively. Following a similar procedure as shown above, one may also obtain from (11a)-(11f) an exact shock solution for unsteady one-dimensional flows expressed explicitly in terms of any desirable shock strength parameter other than $\bar{\rho}$. In the analysis for this case u_n/a takes the place of τ and the given upstream parameters are u_t/a, b_n/a, and b_t/a instead of M_I, k_I, and ψ_I (or s).

(B) Solution in terms of ε

(i) Analytical solution

In actual steady flow problems, the boundary conditions are most often specified in such a way that the flow deflection angle across the shock, θ (hence $\varepsilon = \tan\theta$), is prescribed as the shock strength parameter. The shock angle, β, should then be first determined to locate the shock inclination. To this end, we seek a relation consisting of τ and ε only. This is obtained by elimination of $\bar{\rho}$ from (8a) and (8b); then (see Appendix B for a proper procedure),

$$f(\tau,\varepsilon;M_I,k_I,s;\gamma) = (1+\tau\varepsilon)^3F_1 + (1+\tau\varepsilon)^2(\tau-\varepsilon)F_2 + (1+\tau\varepsilon)(\tau-\varepsilon)^2F_3 + (\tau-\varepsilon)^3F_4 = 0 \qquad (17)$$

where F_1, F_2, F_3, and F_4 do not contain ε and are polynomials of

τ given by

$$F_1(\tau) = k_I^2(\tau-s)^3[(\gamma-1)M_I^2\tau^2(1+s^2) + 2(\tau-s)^2]$$

$$\begin{aligned}
F_2(\tau) = k_I\Big\{&M_I^4\tau^3(1+s^2)\Big\{(\gamma-1)(1+s^2)[2s(1+\tau^2) + (2-\gamma)\tau(1+s\tau)] \\
&- \gamma s(1+s\tau)^2\Big\} + M_I^2\tau(1+s^2)(\tau-s)^2\Big\{k_I[(\gamma+1)s(1+\tau^2) \\
&+ (\gamma-1)^2\tau(1+s\tau)] + 2[2s(1+\tau^2) + (2-\gamma)\tau(1+s\tau)]\Big\} \\
&+ 2(\gamma-1)k_I(1+s\tau)(\tau-s)^4\Big\}
\end{aligned}$$

$$\begin{aligned}
F_3(\tau) = {}&(\gamma-1)M_I^6\tau^4(1+s^2)^2[(\gamma-1)\tau-s] \\
&+ M_I^4\tau^2(1+s^2)\Big\{k_I\Big\{2(\tau-s)[(\gamma+1)s^2(1+\tau^2)-\gamma(\gamma-1)(1+s^2)\tau^2] \\
&+ \gamma[(\gamma-1)\tau-s](1+s\tau)^2\Big\} + 2(1+s^2)\Big\{(\gamma-1)(\tau-s)\tau^2 \\
&+ (1+\tau^2)[(\gamma-1)\tau-s]\Big\}\Big\} + M_I^2\tau(1+s^2)(\tau-s)\Big\{k_I^2\Big\{2\gamma\tau(1+s\tau)^2 \\
&+ (\gamma+1)(\tau-s)[(\gamma-1)\tau(\tau-s)-\gamma(1+s\tau)]\Big\} + 2k_I\Big\{\tau[2(1+s^2)(1+\tau^2) \\
&+ \gamma(1+s\tau)^2 - 2\gamma(\tau-s)^2] - (\gamma+1)(\tau-s)(1+\tau^2)\Big\} + 4\tau(1+s^2)(1+\tau^2)\Big\} \\
&+ 2k_I(\tau-s)^3\Big\{k_I[(\gamma+1)(\tau-s)^2-\gamma(1+s\tau)^2] - 2(1+s^2)(1+\tau^2)\Big\}
\end{aligned}$$

$$\begin{aligned}
F_4(\tau) = (\gamma+1)\Big\{&M_I^6\tau^3(1+s^2)^2[s-(\gamma-1)\tau] \\
&+ M_I^4\tau^2(1+s^2)(\tau-s)\Big\{k_I[\gamma(\tau-s)+\gamma\tau(1+s^2)-2(1+s\tau)(s+\tau)] \\
&- 2\tau(1+s^2)\Big\} + M_I^2\tau(1+s^2)(\tau-s)^2k_I\Big\{k_I[2\tau(1+s\tau)-(\gamma+1)(\tau-s)] \\
&+ 2[\tau-s+\tau(1+s\tau)]\Big\} - 2k_I^2(1+s\tau)(\tau-s)^4\Big\}
\end{aligned}$$

Equation (17) is a polynomial of degree 8 in τ or a cubic equation in ε; it is the most general non-relativistic MHD shock solution relating τ and ε (or β and θ). The analytical structure of the solution is illustrated by its present representation while various limiting solutions derived from (17) can be readily factorized into compact expressions. To write (17) in full as a polynomial either in τ or in ε for the purpose of numerical computations is immediate. Elimination of $\bar{\rho}$ in the auxiliary equations, (8c)-(8f), with the aid of (8b) would yield explicit solutions in terms of τ and ε. One may further eliminate τ in these equations by the use of (17) and obtain solutions in terms of ε only. Rather tedious calculations are involved in these straightforward algebraic manipulations, whereas they are not necessary in actual numerical computations.

We present in the following the shock solutions for several important special cases by making appropriate limits of the parameters in (17) which describes MHD shocks in steady flows. The nature of degeneracy of the solution in each of these limits as viewed from the general solution are revealed in this process. Distinct features characteristic of these limits, especially in the singular cases where simultaneous limits are necessary, are discussed. Some of these limiting solutions have been obtained from other simpler and more restricted formulations in literature. The agreement of all these diversified limiting solutions with known results in each of their respective cases serves also to confirm the correctness of the complicated expression in (17). Alternative forms of each of these special solutions expressed in terms of $(u_n/a,\ u_t/a,\ b_n/a,\ b_t/a)$ which are suitable to describe shocks in unsteady one-dimensional flows, as well as the derivations of the basic properties summarized here, are given in Appendix D.

(a) Aligned field (s = o). This special case of MHD shocks is most extensively treated in the literature. Its solution corresponds to setting $s = \tan \psi_I = 0$ in (17). We get

$$f(\tau,\varepsilon;M_I,k_I,o;\gamma) = \tau^3[(\gamma-1)M_I^2 + 2]A = 0 \tag{18}$$

where

$$\begin{aligned} A = A(\tau,\varepsilon) = {} & k_I\tau(1+t\varepsilon)^2\Big\{k_I\tau(1+\tau\varepsilon) + [(2-\gamma)M_I^2+(\gamma-1)k_I](\tau-\varepsilon)\Big\} \\ & + (M_I^2-k_I)(\tau-\varepsilon)^2\Big\{[(\gamma-1)M_I^2\tau^2+\gamma k_I+2(1+\tau^2)-(\gamma+1)k_I\tau^2](1+\tau\varepsilon) \\ & - (\gamma+1)(M_I^2-k_I)\tau(\tau-\varepsilon)\Big\} . \end{aligned}$$

Thus three finite roots of τ in the case of general magnetic field now degenerate to $\tau = 0$. Nontrivial solutions of τ are given by $A = o$ which is a polynomial of degree 5 in τ or a cubic equation in ε. Writing $A = 0$ in full, one finds that it is in agreement with Cabannes' result in ref. 11.[20]

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[20] There is a misprint in the expression for c in (7) of Ref.11, where the leading term for the coefficient of $\tan^3\theta$ must be $-\frac{\gamma-1}{2}x^4$ instead of $\frac{\gamma-1}{2}x^4$ (in his notations).

(b) Very weak magnetic field ($k_I << 1$). Setting $k_I \rightarrow$ in (17) yields

$$f(\tau,\varepsilon;M_I,0,s;\gamma) = (1+s^2)^2 M_I^2 \tau^2 (\tau-\varepsilon)^2 \left\{ M_I^2[(\gamma-1)\tau-s]+2(\tau-s) \right\} W = 0 \tag{19}$$

where

$$W = W(\tau,\varepsilon) = [(\gamma-1)M_I^2\tau^2+2(1+\tau^2)](1+\tau\varepsilon)-(\gamma+1)M_I^2\tau(\tau-\varepsilon) .$$

Thus, five roots of τ degenerate in this case. Nontrivial solutions are given by W = 0 which agrees with the well-known oblique shock solution in supersonic gasdynamic flows.[21] In this limit all fluid variables are governed by the familiar gasdynamic shock solution as if the very weak magnetic field is absent. On the other hand, the magnetic field depends on the shock strength and the fluid properties. Thus the influence is only "one way." It is also of interest to note that $\bar{B}^2 \geq 1$ always in this case which shows that the shock belongs to fast shocks.

(c) Very strong magnetic field ($k_I >> 1$). Here by very strong magnetic field we mean that the magnetic pressure is much higher than the gas pressure, and hence $k_I >> 1$. Eq. (17) is a quadratic equation in k_I; retaining only terms with k_I^2 yields

$$\begin{aligned} f(\tau,\varepsilon;M_I,k_I>>1,s;\gamma) = k_I^2(\tau-s)(1+\tau^2)^2[(\tau-s)(1+\varepsilon^2) \\ + \gamma(\tau-\varepsilon)(1+\varepsilon s)]S = 0 \end{aligned} \tag{20}$$

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[21] See, e.g., H. W. Liepmann and A. Roshko, Elements of Gasdynamics, John Wiley and Sons, Inc., New York (1957), p.87.

where

$$S = S(\tau,\varepsilon) = M_I^2\tau(1+s^2)[2\tau(s-\varepsilon)+(\gamma+1)(\tau-s)\varepsilon] + 2(\tau-s)^2(\varepsilon-s).$$

Thus six roots of τ degenerate in this case. Nontrivial solutions for the shock are given by $S = 0$ which is a quadratic equation for τ. In this limit $\vec{B}_F = \vec{B}_I$, while the fluid properties depend on $\vec{B}_I$ only through the orientation of the latter. So the influence is also "one way," but in the opposite sense as compared with the case of very weak magnetic field.

(d) Zero flow deflection angle ($\varepsilon = 0$). Setting $\varepsilon = 0$ in (17) yields

$$f(\tau,0;M_I,k_I,s;\gamma) = F_1 + \tau F_2 + \tau^2 F_3 + \tau^4 F_4 = 2Z_1Z_2 = 0 \qquad (21)$$

where

$$Z_1 = Z_1(\tau) = M_I^4\tau^4(1+s^2)-M_I^2\tau^2(1+k_I)(1+s^2)(1+\tau^2)$$
$$+ k_I(1+\tau^2)(\tau-s)^2$$

$$Z_2 = Z_2(\varepsilon) = M_I^2\tau^2(1+s^2)[s-(\gamma-1)\tau]-(\tau-s)\Big\{2\tau^2(1+s^2)$$
$$+k_I s(1+\tau^2)[(\gamma+1)\tau-s]\Big\} .$$

$Z_1 = 0$ is a biquadratic equation in τ; its four solutions give the inclinations of shocks of zero strength across which the state of the gas does not change. It is just the equation for characteristics which can also be derived directly from the system of governing differential equations for the flow.

The characteristics may be regarded as the limiting inclinations of small disturbance waves (or shocks of infinitely weak strength); they may consist of two fast and two slow waves, two slow waves, or four slow waves in general. These are the only solutions for shocks of zero strength. Solutions for weak or moderately strong MHD shocks can be derived readily by perturbations about these solutions.[17]

$Z_2 = 0$ is another biquadratic equation of τ. Its four solutions describe, however, shocks of finite strength in general. Since k_I is associated with τ^4 which is the highest degree term of τ in $Z_2 = 0$, $\tau \rightarrow \infty$ should be one of the solutions as $k_I \rightarrow 0$. This corresponds to the well-known distinguished limiting solution for the strong branch of oblique shocks in supersonic gas-dynamic flows, i.e., $\beta = \frac{\pi}{2}$, $u_t = 0$, and $\varepsilon = 0$. It describes a normal shock and is independent of the upstream Mach number, M_I. We see from $Z_2 = 0$ that in magnetohydrodynamics ($k_I \neq 0$) the inclination of shocks of finite strength at $\varepsilon = 0$ depends on M_I, k_I, and s; they are generally not normal to both upstream and downstream flow directions. This property can also be easily ascertained physically, since the presence of the magnetic field generally yields different tangential momentum on two sides of the shock surface.

It is appropriate at this point to classify the shock adiabatics[22] in the (τ,ε) plane, or equivalently in the (β,θ)

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[22] The term "shock adiabatic" is used here to describe in the general sense for each given upstream state the shock solution relating any two downstream variables (not necessarily $\bar{p}$ and $\bar{\rho}$, e.g., τ and ε are taken here) of which one may be regarded as the shock strength parameter.

plane pertaining to any given M_I, k_I, s, and γ, into two different branches. (Note that only shock adiabatics with $\bar{\rho} \geq 1$ satisfying the entropy condition are considered here.) We define the weak branch to consist of the parts of shock adiabatics which originate from each of the solutions for shocks of zero strength (i.e., the characteristic solutions as discussed above) and extend up to include their respective first local maximum (or minimum) that $\frac{d\varepsilon}{d\tau} = 0$, or equivalently, $\frac{d\theta}{d\beta} = 0$. Thus for any given upstream state, there are generally four separate weak branches describing the fast and slow shocks on (P) and (N) sides of $\vec{u}_I$. All remaining parts of the shock adiabatics are defined to belong to the strong branch, for they describe only shocks of finite strength. Only if the shock solution lies on the weak branch of shock adiabatics can the shock strength be weakened to zero by decreasing $|\theta|$ monotonically toward zero without leaving its own branch. Furthermore, if one restricts consideration to shocks of the weak branch only, there is only one solution of τ (or β) for each given ε (or θ) describing each kind of shock as mentioned above. The shock adiabatics then give unique solutions of shocks and they are well defined. On the other hand, if the shock solution lies on the strong branch, the shock strength cannot be brought to zero within its own branch. It should first pass to the weak branch and then approach from the latter to the solution for zero strength shocks. During this process, double or multiple values of τ

(or β) for a certain range of ε (or θ) are encountered. Apparently the weak branches of shock adiabatics are more interesting physically; they probably describe most of the actual MHD shocks in nature whenever they are subjected to normal boundary conditions. Shocks corresponding to the strong branch of shock adiabatics may occur only under unusual boundary conditions. In the limit of very weak magnetic fields, the above definitions are in agreement with the familiar definitions for weak and strong branches of oblique shocks in supersonic gasdynamic flows.

(e) Normal upstream magnetic field ($B_{I,t} = 0$, or $s = -\tau^{-1}$). $\vec{B}_I \times \vec{e}_n = 0$ may be regarded here as the limit of $\vec{B}_I \times \vec{e}_n \to 0$, so that $\vec{e}_t$ and $\vec{e}_z$ are well defined as usual. This case appears to be a rather special situation in steady flows where the orientation of the shock line as determined by a certain shock strength parameter specified downstream (i.e., ε here) just happens to be perpendicular to the given upstream magnetic field direction. It yields two important classes of shock solutions; one describes pure gasdynamic shocks and the other describes complete switch-on shocks.

$B_{I,t} = 0$ means $s = -\tau^{-1}$; setting $s = -\tau^{-1}$ in (17) yields

$$f(\tau,\varepsilon;M_I,k_I,-\tau^{-1};\gamma) = \tau^{-5}(1+\tau^2)^2 N_1 N_2 = 0 \qquad (22)$$

where

$$N_1 = N_1(\tau,\varepsilon) = [(\gamma-1)M_I^2\tau^2 + 2(1+\tau^2)](1+\tau\varepsilon) - (\gamma+1)M_I^2\tau(\tau-\varepsilon)$$

$$N_2 = N_2(\tau,\varepsilon) = [M_I^2\tau(\tau-\varepsilon) - k_I(1+\tau^2)(1+\tau\varepsilon)]^2$$
$$+ (\tau-\varepsilon)^2[M_I^2\tau^2 - k_I(1+\tau^2)][2(1+\tau^2)+(\gamma-1)M_I^2\tau^2 - k_I(\gamma+1)(1+\tau^2)]$$

A comparison of $N_1 = 0$ with $W = 0$ in (19) shows that it is just the gasdynamic shock solution. $\vec{B}_F = \vec{B}_I$ across this shock. The changes of other physical variables are determined by the familiar solution for gasdynamic oblique shock as if $\vec{B}_I$ were absent. Setting $\tau = -s^{-1}$ in $N_1 = 0$ yields

$$\varepsilon = \frac{2s(1+s^2-M_I^2)}{M_I^2[\gamma-1+(\gamma+1)s^2] + 2(1+s^2)} . \qquad (23)$$

Eq. (23) is independent of the magnetic field strength since k_I does not enter in the equation. Thus for any given upstream state with M_I, s and γ known if ε satisfies $N_1 = 0$ in (22), one of the shock solutions is a pure gasdynamic shock. This type of shock is sometimes called a parallel shock in the sense that $\vec{B}_I \parallel \vec{e}_n$ and $\vec{B}_F \parallel \vec{e}_n$.

$N_2 = 0$ corresponds to $1+\tau s \to 0$ and $M_I^2 \to \bar{\rho}k_I(1+s^2)$ simultaneously. This solution exists only if $k_I > 1$ and $M_I^2 > k_I(1+s^2)$. It represents a complete switch-on shock behind which a finite tangential magnetic field, $\vec{B}_{F,t}$, appears, and obviously it belongs to fast shocks. The solutions in (8a)-(8f) are singular in this case, and an appropriate limiting process is necessary. The magnitude of $\vec{B}_{F,t}$ depends only on upstream conditions and is given by

$$B_{F,t}^2 = B_I^2\left[\frac{M_I^2}{k_I(1+s^2)} - 1\right]\left\{2\left(1 - \frac{1}{k_I}\right) - (\gamma-1)\left[\frac{M_I^2}{k_I(1+s^2)} - 1\right]\right\} \qquad (24)$$

Setting $\tau = -s^{-1}$ in $N_2 = 0$ yields the following quadratic equation in ε:

$$[M_I^2 s(1+s\varepsilon) - k_I(1+s^2)(s-\varepsilon)]^2 + (1+s\varepsilon)^2[M_I^2 - k_I(1+s^2)][(\gamma-1)M_I^2 - k_I(\gamma+1)(1+s^2) + 2(1+s^2)] = 0\ . \qquad (25)$$

Thus for any given M_I, k_I, s and γ, a complete switch-on MHD shock arises if ε satisfies (25).

The conditions specified in both (23) and (25) are necessary but not sufficient, since only one of several possible solutions of shocks corresponding to the known upstream state and ε gives rise to the specific type of shock discussed here.

Note that an analogous situation occurs if $B_{I,t} \neq 0$ but $B_{F,t} = 0$; it corresponds to a complete switch-off shock and is necessarily a slow shock.

(f) Tangential magnetic field ($B_{I,n} = 0$ or $\tau = s$). $B_{I,n} = 0$ implies $B_{F,n} = 0$ as required by (1a). The corresponding shock is sometimes called a perpendicular shock in the sense that the magnetic fields on both sides of the shock are perpendicular to $\vec{e}_n$. $B_{I,n} = 0$ corresponds to $s = \tau$; then (17) becomes

$$f(\tau,\varepsilon;M_I,k_I,\tau;\gamma) = (2-\gamma)M_I^4\tau^3(1+\tau^2)^2(\tau-\varepsilon)T = 0 \qquad (26)$$

where

$$T = T(T,\varepsilon) = (\gamma-2)k_I\tau(1+\tau^2)(1+\tau\varepsilon)^2$$
$$-[(\gamma-1)M_I^2\tau^2 + (\gamma k_I+2)(1+\tau^2)](1+\tau\varepsilon)(\tau-\varepsilon)+(\gamma+1)M_I^2\tau(\tau-\varepsilon)^2 \quad . \tag{26}$$

Nontrivial solutions are given by $T(\tau,\varepsilon) = 0$ which is a polynomial of degree five in τ or a quadratic equation in ε. Setting $\tau = s$ in $T = 0$ yields an equation from which ε can be solved in terms of the upstream parameters. Although this situation appears to be rather special in steady flows, $T = 0$ actually also describes MHD shock solutions in transverse flows. To see this let us consider general two-dimensional steady flows; transverse flows correspond to $\vec{e}_G \parallel \vec{B}_{I,t}$ and $\vec{e}_G \parallel \vec{B}_{F,t}$ where $\vec{e}_G$ is the unit vector in the ignorable (or transverse) direction. Thus $\vec{e}_G = \vec{e}_t$ and the (Z,n) plane is the plane in which spatial variation takes place, whereas the shock surface is, by definition, represented by the (t,Z) plane. It follows from (1c) that at $B_{I,n} = B_{F,n} = 0$, $\vec{U}_{I,T} = \vec{U}_{F,T}$; hence both u_t and U_Z are conserved across the shock. Without loss of generality we may consider $u_{I,n} \neq 0$, $u_{I,t} \neq 0$ but $U_{I,Z} = 0$ which is just the case presently considered, although in most transverse flows one is concerned, instead, with the situation that $u_{I,n} \neq 0$, $U_{I,Z} \neq 0$ but $u_{I,t} = 0$. Familiar limiting solutions for cases of very weak magnetic field ($k_I \rightarrow 0$) and very weak shocks ($\varepsilon \rightarrow 0$) can be obtained readily from $T = 0$ in (26). For the case of very strong magnetic field ($k_I \rightarrow \infty$), $T = 0$ reduces to

$$(2-\gamma)\varepsilon\tau^2 + 2\tau - \gamma\varepsilon = 0 . \tag{27}$$

Eq. (27) is a quadratic equation of τ whose solution is independent of M_I^2. Note that (27) cannot be obtained from (20) by taking the limit of $\tau \rightarrow s$, because $T = 0$ is a linear rather than a quadratic equation of k_I. Hence the limiting processes of $k_I \rightarrow \infty$ and $\tau \rightarrow s$ taken from (17) are not interchangeable.

(g) Non-characteristic solutions with $\bar{\rho} = 1$. For each given upstream state with M_I, k_I, and s being perturbed, $\bar{\rho} = 1$ occurs not only at the four characteristic solutions with $\varepsilon = 0$ as determined by $Z_1 = 0$ in (21), but also at two other solutions of τ with $\varepsilon \neq 0$. The latter solutions appear in the simultaneous limit of $\bar{\rho} \rightarrow 1$ and $M_I^2\tau^2(1+s^2) \rightarrow k_I(\tau-s)^2$ (i.e., $u_n^2 \rightarrow b_n^2$); they are associated only with slow shocks across which $\bar{P} = \bar{B} = 1$, but the tangential magnetic field component reverses its direction and $\vec{u}_F \neq \vec{u}_I$ in general. Therefore they do not represent shocks of zero strength and are entirely different from the characteristic solutions. For this case τ is determined by

$$[M_I^2(1+s^2) - k_I]\tau^2 + 2k_I s\tau - k_I s^2 = 0 \tag{28a}$$

which yields two solutions. We have also

$$\varepsilon = \frac{2\tau^2(1+s\tau)}{\tau(1-\tau^2) + s(1+3\tau^2)} \tag{28b}$$

from which ε can be easily determined after τ is solved from (28a). These two new solutions, together with the four characteristic solutions, provide the entire six end points of the postions of shock adiabatics in the (τ,ε) plane which describe shocks satisfying the entropy condition and are of direct physical interest. Note that γ does not enter in all these six solutions, and hence the end points are independent of the particular kind of gas in consideration.

(ii) Numerical results

To illustrate the implications of the analytical solution in (17), numerical results are obtained by the use of an IBM-7094 computer. They are illustrated in Figs. 2 and 3.

$\pi \geq \beta \geq 0$ always. Also it suffices to consider θ ranging from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ since (17) is periodic in θ with the period π. Without loss of generality we may restrict ψ_I to range from o to $\frac{\pi}{2}$, whereas the solution for $0 \geq \psi_I \geq -\frac{\pi}{2}$ can be readily obtained from the former by the use of the transformation specified in (9). In Figs. 2 and 3 the solid lines describe the portions of the shock adiabatic satisfying the entropy condition $\bar{S} \geq 1$ (or $\bar{\rho} \geq 1$), whereas the dashed lines describe those for shocks across which $\bar{S} < 1$ (or $\bar{\rho} < 1$).

The locations of the fast and slow characteristics as well as the non-characteristic solutions with $\bar{\rho} = 1$, are indicated by ω_+,

ω_-, and r respectively. For any given upstream state and gas with M_I, k_I, and s together with γ known, the corresponding shock adiabatic on the (β,θ) plane is obtained by solving the polynomial in (17). The cubic equation in ε, with the values of τ being prescribed, is first solved; the resulting solutions are subsequently checked against that obtained by solving the polynomial of degree eight in τ, with the values of ε being prescribed. The presentation of all possible solutions covering entire ranges of all four independent parameters is obviously out of the question. An attempt is made here to illustrate the essential features of the shock adiabatics at certain representative values of these parameters. One may then infer from these solutions the behavior of the shock adiabatics at other values of the governing parameters.

γ may take values only within a very limited range; its variation does not alter significantly the qualitative nature of the solution in general. Moreover, all end points of the shock adiabatics satisfying the entropy condition (i.e., the solid lines in Figs. 2 and 3) are, in fact, independent of γ as seen from $Z_1 = 0$ in (21) and (28). We consider $\gamma = \frac{5}{3}$ describing monatomic gases in all the numerical computations.

Figs. 2a-f describe the shock adiabatics at $M_I = 3.0$ and $\psi_I = 30^\circ$ with only k_I being different in various graphs. Thus the variation of the solution due to the change of the strength of the magnetic field with its direction being fixed at a finite angle with respect to the flow direction is illustrated.

Guided by the familiar gasdynamic solution at $k_I = 0$ for which the shock adiabatic is antisymmetric with respect to $\beta = \frac{\pi}{2}$ and is independent of ψ_I, we may observe from Fig. 2a the initial emergence of slow shocks in the presence of a weak magnetic field at $k_I = 0.5$. Figs. 2b-f show the shock adiabatics as k_I increases gradually from 3.0 to 40.0. We obtain the equation for characteristic locus[23] that $M_I = 3.0$ and $\psi_I = 30^\circ$; the upstream flow field is either hyperbolic with the existence of two fast and two slow characteristics, or "hyperliptic" with the existence of only two slow characteristics, depending on whether k_I is greater or less than 8.77 respectively. It is clearly manifested in Figs. 2d and 2e that the fast characteristics dieappear as k_I increases from 8.0 to 10.0. Figs. 2e and 2f show that the qualitative behavior of the shock adiabatic remains unaltered as k_I increases from 10.0 even up to 40.0 for the case of very strong magnetic field.

For any fixed k_I and any fixed sufficiently small ψ_I, let us first consider the case that M_I is so large that the upstream state is hyperbolic; that is, there exist two fast and two slow characteristics (i.e., if $-\vec{u}_I$ is outside of the characteristic locus). As M_I decreases continuously from that value, it can be seen from the characteristic locus pertaining to k_I that the upstream flow field first becomes "hyperliptic" with the existence of two slow characteristics only; it subsequently becomes hyperbolic again with the existence of four slow characteristics.

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[23] Y. M. Lynn, Phys. Fluids 5, 626 (1962).

As M_I decreases further toward zero, the flow field becomes "hyperliptic" again with the existence of two slow characteristics only. Naturally it would be very interesting to investigate here the variation of the shock adiabatics as the upstream state traverses through each of the above mentioned flow regimes. Since the inner cusped characteristic locus corresponding to slow waves is usually situated near $\psi_I = 0$, the upstream state of flow at which there exist four slow characteristics occurs only when ψ_I is sufficiently small. It is also of interest to investigate the degenerate case of $\psi_I = 0$ describing aligned flows for which the shock adiabatics should be antisymmetric with respect to $\beta = \frac{\pi}{2}$. Moreover, one would like to observe the gradual approach of the shock adiabatics to the degenerate aligned field solutions as ψ_I decreases continuously from finite values to zero. The shock adiabatics for all cases enumerated above are illustrated in Figs. 3. Fig. 3a describes the positions of the terminals of $-\vec{u}_I$ relative to its corresponding characteristic locus for each of the cases considered in Figs. 3b-3n. Since the general character of the solutions is fundamentally different at $k_I > 1$ and $k_I < 1$ when ψ_I is small, the shock adiabatics at $k_I = 0.5$ and 2.0 are considered, and they are shown in Figs. 3b-3g and Figs. 3h-3n respectively.

It is shown in Figs. 3f and 3ℓ and also in Figs. 3g and 3n that in aligned field flows there exists only one type of shocks always; it may consist of either two downstream-facing fast shocks or two upstream-facing slow shocks, always symmetric with respect to the flow direction. An anomalous situation occurs when M_I, k_I and ψ_I are $(1.20, 2.0, 0^\circ)$ respectively. The corresponding terminal of $-\vec{u}_I$ is at m as shown in Fig. 3a, and hence the flow field is "hyperliptic" and no real characteristics exist except two degenerate ones coinciding with the streamlines. Nevertheless the shock solutions satisfying the entropy-increase condition do exist as shown in Fig. 3m on which $\bar{\rho}$ is actually found to be always bounded away from 1. Clearly these anomalous shock solutions cannot be evolved continuously from that for zero-strength shocks (i.e., characteristics) with $\theta = 0$ and $\bar{\rho} = 1$. Perhaps they describe physically unrealistic non-evolutionary shocks to be excluded after the evolutionary conditions are taken into account (see further discussions in Section IV). On the other hand, no real shock adiabatics satisfying the entropy-increase condition exist when the upstream state belongs to other aligned flow regimes, e.g., when M_I, k_I, and ψ_I are $(0.85, 0.5, 0^\circ)$, $(0.40, 0.5, 0^\circ)$, and $(0.40, 2.0, 0^\circ)$ respectively. Figs. 2 and 3 show that all shock adiabatics with $\bar{\rho} \geq 1$ satisfy (6b) automatically. The portion which does not satisfy (6b) actually corresponds to $\bar{\rho}_F < 0$ (or $\bar{\rho} < 0$) which is obviously unphysical and has been excluded from our previous derivation of (6b) from (5a).

(C) Solution in terms of ε_B (= $\tan(\psi_F - \psi_I)$)

MHD shocks may be generated not only mechanically but also by electromagnetic means, for example, by a magnetic dipole or a current sheet situated downstream. Thus the deflection angle of the magnetic field vector across the shock, $\theta_B = \psi_F - \psi_I$, is sometimes prescribed by the boundary condition in flow problems as a natural shock strength parameter.

To find the solution in terms of θ_B, we first find the solution relating τ and ϕ which can be obtained by eliminating $\bar{\rho}$ from (8a) and (8c). Thus we get

$$g(\tau,\phi;M_I,k_I,s;\gamma) = (1+\tau\phi)^3 G_1 + (1+\tau\phi)^2(\tau-\phi)G_2$$
$$+ (1+\tau\phi)(\tau-\phi)^2 G_3 + (\tau-\phi)^3 G_4 = 0 , \qquad (29)$$

where G_1, G_2, G_3, and G_4 do not contain ϕ and are polynomials of τ given by

$$G_1(\tau) = k_I^2(\tau-s)^5$$

$$G_2(\tau) = k_I(\tau-s)^2(1+s\tau)[(2-\gamma)M_I^2\tau^2(1+s^2)+(\gamma-1)k_I(\tau-s)^2]$$

$$G_3(\tau) = (\tau-s)[M_I^2\tau^2(1+s^2)-k_I(\tau-s)^2]\Big\{(1+s^2)[(\gamma-1)M_I^2\tau^2+2(1+\tau^2)]$$
$$+ \gamma k_I(1+s\tau)^2-(\gamma+1)k_I(\tau-s)^2\Big\}$$

$$G_4(\tau) = -(\gamma+1)(1+s\tau)[M_I^2\tau^2(1+s^2)-k_I(\tau-s)^2]^2 .$$

Define $\varepsilon_B = \tan\theta_B = \tan(\psi_F-\psi_I)$; then $\phi = \tan\psi_F \dfrac{\varepsilon_B+s}{1-\varepsilon_B s}$.

Substitution of this into (29) yields

$$g_B(\tau,\varepsilon_B;M_I,k_I,s;\gamma) \tag{30}$$

$$= [1+s\tau+\varepsilon_B(\tau-s)]^3G_1 + [1+s\tau+\varepsilon_B(\tau-s)]^2[\tau-s-\varepsilon_B(1+s\tau)]G_2$$

$$+[1+s\tau+\varepsilon_B(\tau-s)][\tau-s-\varepsilon_B(1+s\tau)]^2G_3+[\tau-s-\varepsilon_B(1+s\tau)]^3G_4 = 0$$

This is a polynomial of degree eight in τ and a cubic equation for ε_B from which the root of τ can be determined for each prescribed ε_B. For flows aligned upstream $s = 0$ and (30) becomes the same equation as $A = 0$ in (18), provided ε_B is replaced by ε; hence $\theta_B = \theta$ and it indicates the well-known fact that the flow remains aligned downstream.

III. Connections with Other Existing Solutions

Various explicit analytical solutions for MHD shocks have appeared in the literature. In this section the connections of some of the other major results with our general solution are illustrated and discussed. To facilitate direct comparisons the notations used in other works are superimposed with a tilde whenever they are quoted below.

(A) In arbitrary magnetic field

(i) In Cole's paper[13] a one-dimensional MHD shock propagating with a uniform velocity $\tilde{c}$ into an ideal gas otherwise at rest is considered to be generated by the impulsive motion of a piston with a uniform normal velocity $\tilde{u}_F$ into the gas behind. In a coordinate system fixed to the shock front, this corresponds to the special case of normal incidence, $u_t = 0$, discussed in Section II and Appendix C. The correspondence of notations is $u_n = \tilde{c}$, $a = \tilde{a}_I$, $b_n = -\tilde{b}_x$, $b_t = \tilde{b}_y$, $u_{n,F} = \tilde{c}-\tilde{u}_F$, $u_{t,F} = \tilde{v}_F$, so that

$$\varepsilon = \frac{\tilde{v}_F}{\tilde{c}-\tilde{u}_F} = \frac{-\tilde{u}_F\tilde{b}_x\tilde{b}_y}{(\tilde{c}-\tilde{u}_F)(\tilde{c}^2-\tilde{c}\tilde{u}_F-\tilde{b}_x^2)} \ .$$

Then $I = 0$ in Eq. (C1) becomes

$$c_1 c_2 = 0 \tag{31}$$

where

$$\begin{aligned} c_1 = {} & (\tilde{c}^2-\tilde{b}_x^2)(\tilde{c}-\tilde{u}_F)^2[2\tilde{u}_F(\tilde{c}^2-\tilde{a}_I^2)-(\gamma+1)\tilde{c}(\tilde{c}^2-\tilde{b}_x^2)] \\ & - \tilde{u}_F\tilde{b}_y^2\,\tilde{c}[\gamma\tilde{c}^3-(\gamma+2)\tilde{u}_F\tilde{c}^2 + (2\tilde{u}_F^2-\gamma\tilde{b}_x^2)\tilde{c} + (\gamma+1)\tilde{u}_F\tilde{b}_x^2] \\ c_2 = {} & [2(\tilde{c}^2-\tilde{a}_I^2)-(\gamma+1)\tilde{u}_F\tilde{c}](\tilde{c}^2-\tilde{u}_F\tilde{c}-\tilde{b}_x^2)^2 \\ & - \tilde{c}\tilde{b}_y^2[2\tilde{c}^3-(\gamma+2)\tilde{u}_F\tilde{c}^2 + (\gamma\tilde{u}_F^2-2\tilde{b}_x^2)\tilde{c} + (\gamma+1)\tilde{u}_F\tilde{b}_x^2] \ . \end{aligned}$$

Here $\tilde{u}_F$ is considered as the prescribed shock strength parameter, and the shock speed $\tilde{c}$ can be determined from (31). $c_1 = 0$ gives $\tilde{c}^3(\tilde{c}-\tilde{b}_x^2)^2 = 0$ at $\tilde{u}_F = 0$; it does not describe the compressive shock solution and must be discarded. $c_2 =$ agrees precisely with Cole's solution.[24]

(ii) Kiselev has obtained the solution for general MHD shocks as a cubic equation for $\bar{B}_t = B_{F,t}/B_{I,t}$, together with u_n and the upstream parameters. We solve for $\bar{\rho}$ from (13) and get

$$\bar{\rho} = \frac{u_n^2 \bar{B}_t}{u_n^2 + b_n^2(\bar{B}_t - 1)} \quad . \tag{32}$$

Substitution of (32) into (11a) yields

$$\bar{B}_t^3 + [1-(2-\gamma)(1-\frac{u_n^2}{b_n^2})]\bar{B}_t^2 + (1-\frac{u_n^2}{b_n^2})(b_n^2 - \frac{2a^2+(\gamma-1)u_n^2+\gamma b_t^2}{\gamma+1})\frac{\gamma+1}{b_t^2}\,\bar{B}_t$$
$$- (\gamma+1)\,\frac{b_n^2}{b_t^2}\,(1 - \frac{u_n^2}{b_n^2})^2 = 0 \tag{33}$$

which is in agreement with Kiselev's solution[25] with the

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[24] Cf. Eq. (2)-(7) in ref. 13.

[25] Cf. (6) in ref. 14 or (1) in ref. 15. Unfortunately, both of these equations contain a misprint in their last term for which the correct expression in terms of original notations should read

$$-(k+1)V_{1x}^2 h_{1y}\rho_1(1 - \frac{u_{1x}^2}{V_{1x}^2})^2 \quad .$$

correspondence of notations being $\gamma = \tilde{k}$, $u_n = \tilde{u}_{1x}$, $b_n = \tilde{V}_{1x}$, $B_{F,t}/\sqrt{\mu} = \tilde{h}_{2y}$, $B_{I,t}/\sqrt{\mu} = \tilde{h}_{1y}$, and $(2a^2+(\gamma-1)u_n^2+\gamma b_t^2)/\gamma+1 = \tilde{a}_{1Mx}^2$. The solution for aligned flows may then be easily obtained from (33).

(iii) An equivalent solution to (11a) was obtained from a different approach by Anderson. Dividing (11a) by b_n^6 and expressing it in terms of the notations, $u_n/b_n = \tilde{M}_{A1}$, $b_t/b_n = \tilde{M}_{y1}$, $a/b_n = \tilde{M}_{a1}$, and $\bar{\rho} = \tilde{\beta}$, one gets precisely Anderson's solution.[26] However, no explicit solutions for other downstream variables are available there.

(B) Aligned flows

The special solution for aligned flows has been investigated by many authors. The equation for $\bar{\rho}$ in (11a) now takes a simple form as shown in (D2). Analytical solution for this case was first derived by Helfer.[5] A different and very comprehensive study was later given by Bazer and Ericson.[6-8] The explicit solution expressed in terms of β and θ suitable to describe shocks in steady flows was presented by Cabannes.[11] Its connection with our general solution has been discussed in IIB-i-a.

(i) The correspondence of Helfer's parameters with ours is $\tilde{Q} = (\gamma\kappa/2)(u^2/a^2)$, $\tilde{s} = u_t/u_n$, $\tilde{A} = 1 - 1/\kappa\bar{\rho}$ and $\tilde{x} = \tilde{s}_2/\tilde{s}_1 = ((\tilde{A}-1)\bar{\rho}+1)/\tilde{A}$; one may solve from above and get $\bar{\rho} = (1-\tilde{A}\tilde{x})(1-\tilde{A})$, $\kappa = 1/1-\tilde{A}\tilde{x}$, $u_n^2/a^2 = (2\tilde{Q}(1-\tilde{A}\tilde{x}))/\gamma(1+\tilde{s}^2)$ and $u_t^2/a^2 = (2\tilde{Q}\tilde{s}^2(1-\tilde{A}\tilde{x}))/\gamma(1+\tilde{s}^2)$. Substitution of these into (D2) yields Helfer's solution[27] which is a quadratic equation of $\tilde{A}$ expressed in terms of the parameters $\tilde{Q}$, $\tilde{s}$, and $\tilde{x}$. Here the reducibility of the cubic equation of $\bar{\rho}$ to a quadratic equation of $\tilde{A}$ is essentially due to the fact that $\tilde{x}$ is regarded as a prescribed parameter, so that $\frac{\bar{\rho}-1}{\kappa\bar{\rho}-1} = 1-\tilde{x}$ is then given and $\bar{\rho}-\tilde{x}$ is inversely proportional to $1-\tilde{A}$.

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[26] Cf. (2.43) on p. 21 in ref. 16.

(ii) The correspondence of the parameters used by Bazer and Ericson with ours is $\cos\tilde{\omega} = \cos\beta = u_t/u$, $\tilde{s}_o = a^2/b^2 = a^2/\kappa u^2$, $\tilde{h} = \varepsilon/(\sin\tilde{\omega} - \varepsilon\cos\tilde{\omega})$ and $\tilde{\bar{\eta}} = \bar{\rho}-1$. The equation for ε in (D3) becomes

$$\varepsilon = \frac{\tilde{\bar{\eta}}\sin\tilde{\omega}\cos\tilde{\omega}}{1+\tilde{\bar{\eta}}\cos^2\tilde{\omega} - \kappa(\tilde{\bar{\eta}}+1)} \quad . \tag{34}$$

Eliminating ε in the equation for $\tilde{h}$ by the use of (34), one solves for κ as

$$\kappa = \frac{\tilde{h} - \tilde{\bar{\eta}}\cos\tilde{\omega}}{\tilde{h}(\tilde{\bar{\eta}}+1)} \quad . \tag{35}$$

Substitution of (35), $a^2/u^2 = \kappa\tilde{s}_o = (\tilde{s}_o(\tilde{h}-\tilde{\bar{\eta}}\cos\tilde{\omega}))/\tilde{h}(\tilde{\bar{\eta}}+1)$,

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[27] Cf. (59) in ref. 5.

$u_n/u = \sin\tilde{\omega}$, and $u_t/u = \cos\tilde{\omega}$ into (D2) yields a quadratic equation of $\tilde{\bar{\eta}}$ from which $\tilde{\bar{\eta}}$ can be solved explicitly in terms of $\tilde{h}$, $\tilde{s}_0$, and $\tilde{\omega}$. This is the main result of Bazer and Ericson.[28] Other solutions expressed in terms of $\tilde{h}$, $\tilde{s}_0$, and $\tilde{\omega}$ follow directly. The crucial step of the reduction of a cubic to a quadratic equation of $\bar{\rho}$ (= $\tilde{\bar{\eta}}$-1) lies in the fact that $\frac{\kappa\bar{\rho}-1}{\bar{\rho}-1}$ (= $\frac{\kappa(\bar{\eta}+1)-1}{\tilde{\bar{\eta}}}$) = $-\frac{\cos\tilde{\omega}}{\tilde{h}}$ and it is known from the prescribed parameters. Then (D2) becomes a quadratic equation of $\bar{\rho}$ (or $\tilde{\bar{\eta}}$) since $\kappa\bar{\rho}$ depends linearly on $\bar{\rho}$ (or $\tilde{\bar{\eta}}$) while all other parameters are given.

IV. Dynamical Stability Criteria

In previous sections we were concerned with the transition solution across a general MHD shock with the entropy condition duly taken into account. The fulfillment of the entropy condition alone is necessary but not sufficient to warrant a physically realistic shock. Any realistic shock should also be stable against small disturbances. These dynamical stability criteria were overlooked in a number of works on MHD shocks, perhaps largely because their counterpart in gasdynamics fails to introduce any additional restriction on the shock transition solution other than that already required by the entropy condition. For MHD shocks it has been shown that in the special case of aligned flows the consideration of dynamical stability criteria contributes to exclude a substantial portion of shock adiabatics satisfying the entropy condition.[12] The corresponding criteria for MHD shocks in general nonaligned flows have not been treated up to the present. In view of their underlying importance we present in the following a formulation of the dynamical stability criteria for general MHD shocks and illustrate the present status of knowledge of these criteria, as well as their essential inadequacies. However, no analytical results are attempted here, for the general problem itself is complex enough to merit a separate and extensive investigation.

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[28] Cf. (19) in ref. 6 or (9a) in ref. 8.

Dynamical stability criteria are usually considered to consist of two types. Firstly, any solution for a realistic shock should possess a neighborhood solution such that a small perturbation of the shock arising from arbitrary disturbances on it should result in only a slight change of the shock solution. Mathematically this corresponds to the requirement of the continuity of the solution of a realistic shock in response to a continuous change of boundary data; the existence of a slightly perturbed neighborhood solution is then guaranteed. This condition is often termed an evolutionary condition or stability against break up. Secondly, in addition to the fulfillment of the evolutionary condition, a realistic shock should also be stable against time-dependent small disturbances so that the temporal development of the amplitudes of the perturbed physical quantities should not grow indefinitely as time increases. Both of these criteria can be analyzed in general terms for a shock in an unbounded medium without referring to other local boundary conditions. It is usually an easy matter to determine the evolutionary condition, since it demands only the correct formulation of a well-posed algebraic system of equations governing the small perturbations of physical variables pertaining to MHD shocks. To this end, a simple counting of the number of possible diverging waves propagating away from the shock surface for all possible flow situations, followed by an examination of whether this number being equal to the number of necessary boundary conditions would suffice. It should be emphasized that the evolutionary condition for MHD shocks in unsteady one-dimensional flows is not to be confused with that in steady two-dimensional flows. The characteristics (i.e., the small disturbance waves) in steady two-dimensional flows do not propagate always in a preferred fixed direction in space as they do in one-dimensional unsteady flows; consequently the envelope of the fronts of various normal wave speeds, rather than the fronts themselves, plays a central role of classifying different types of flow situations. In this respect, the characteristic locus,[4,23] rather than the normal wave speeds, should be considered in investigating the evolutionary condition in steady two-dimensional flows. On the other hand, the determination of stability against time-dependent disturbances requires the examination of the sign of the temporal growth rate of various amplitudes for which the system of governing differential equations describing the flow has to be analyzed.

Schematically, the entire shock solution can be summarized as follows. Let us denote, for MHD shocks in either one-dimensional unsteady flow or two-dimensional steady flow with prescribed γ and upstream parameters:

S_A = all transition solutions of MHD shocks satisfying the conservation laws described by (1a)-(1f).

S_s = the closed subset of S_A which satisfies the entropy condition, $\bar{S} \geq 1$ (or $\bar{\rho} \geq 1$).

S_E = the closed subset of S_A satisfying the evolutionary condition.

S_T = the closed subset of S_A which is stable against arbitrary time-dependent small disturbances.

S_R = all solutions which describe physically realistic MHD shocks.

Clearly S_R is our main concern and is given by

$$S_R = S_s \cap S_E \cap S_T \; ; \tag{36}$$

only S_A and S_s are studied in this paper. Analytical solutions of S_A are presented in Section II. In Figs. 2 and 3 the solid lines describe S_s, whereas the complement of S_s relative to S_A, $S_s = S_A' - S_s$, is described by the dashed lines.

Evolutionary conditions for MHD shocks in one-dimensional unsteady flows with arbitrarily oriented magnetic fields were first considered in refs. 29, 30, and 31 for normal and oblique incident wave trains, respectively. The results obtained there were subsequently applied to the analytical solution of one-dimensional MHD shocks in aligned flows.[12,32] In this way the conclusion that the evolutionary condition imposes one extra restriction, viz., $\bar{B}_t \geq 0$ (applicable to slow shocks only), in addition to the entropy condition is valid only for shocks in the degenerate case of aligned flows. In fact, many elegant analytical properties derived for MHD shocks in aligned flows[6,7] no longer hold in general nonaligned flows. Naturally the consequences of the evolutionary condition for the solutions of shocks in these two flows are different in general. We may note that the evolutionary condition for MHD shocks in steady flows has not been treated so far.

It is obvious that $S_E \supset S_T$ always, because S_T presupposes

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29. A. I. Akhiezer, G. J. Liubarskii, and R. V. Polovin, Sov. Phys., JETP 8, 507 (1959).

30. V. M. Kontorovich, Sov. Phys. JETP 8 851 (1959).

31. S. I. Syrovatskii, Sov. Phys., JETP 8, 1024 (1959).

32. A. Jeffrey and T. Taniuti, Non-linear Wave Propagation, (Academic Press, 1964), p. 218

the existence of a slightly perturbed shock solution which belongs to S_E. Then (36) becomes

$$S_R = S_s \cap S_T \ . \tag{37}$$

Hence, so long as S_R alone is desirable, it suffices to find S_T in addition to S_s. The special case of S_T for a normal incident Alfvén wave on a one-dimensional MHD shock was given by Anderson.[16] S_T for general MHD shocks with $B_n = 0$ and $B_t = 0$, respectively, was studied by Gardner and Kruskal.[33] The general case of $B_n \neq 0$ and $B_t \neq 0$ is still an open problem.

Clearly, the dynamical stability criteria serve further to exclude certain parts of the shock adiabatics described by solid lines in Figs. 2 and 3; presumbaly most of the mysteriously behaving portions could be discarded. New analytical properties of MHD shocks in nonaligned flows must also be investigated. They could then be used together with the deduced results according to dynamical stability criteria to provide several conclusive and fundamental properties of general MHD shocks. These would enable a substantial refinement of the somewhat crude basis for the classification of fast and slow shocks considered in Section II. The conditions for the connection of various types of MHD shocks could then be derived.

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[33] C. S. Gardner and M. D. Kruskal, Phys. Fluids 7, 700 (1964).

V. Discussions

Only τ (or u_n) as a function of the shock strength parameter is presented in detail in Section II and Appendix D; all other downstream variables follow readily by the use of auxiliary equations. In steady flows the determination of the maximum deflection angle of streamline, θ_{max}, across each kind of MHD shocks as a function of the given upstream parameters and γ yields directly the criteria for the detachment of MHD shocks. In principle, $\varepsilon_{max} = \tan \theta_{max}$ can be obtained simply by eliminating τ from $f(\tau,\varepsilon) = 0$ and $f_\tau = 0$ (since $d\varepsilon/d\tau = - f_\tau/f_\varepsilon = 0$). Several solutions of θ_{max} exist in general, only a few of which pertain to physically realistic MHD shocks (fast or slow). The analytical derivations of θ_{max} are quite involved, whereas numerical calculations are more convenient and preferable.

The shock thickness is infinitesimal, and it is always much smaller than the curvature of any macroscopic shock surface. Consequently, general curved MHD shocks in steady three-dimensional flows can always be approximated locally by a planar surface for which the present solution can be used.

The solution for general MHD shocks includes a full description of the interplay of several characterizing physical mechanisms; its complicated form is hardly surprising, but solutions for many different limiting cases can be derived readily from it. Final quantitative results can be obtained conveniently only with the aid of an electronic computer. In this regard, a subroutine subprogram (e.g., an IBM binary deck) for the shock solution giving values of all downstream variables with the values of upstream parameters, γ, and the shock strength parameter to be fed in, is very convenient in solving MHD flow problems, while a table containing numerical data with full ranges of all basic parameters covered is not feasible here.[34]

After a complete understanding of the dynamical stability criteria for general MHD shocks is acquired, one may use the present shock solution with assured confidence to analyze quantitatively many MHD flow problems which occur in several branches of physics of great current interest, e.g., the controlled fusion process, geophysical and astrophysical problems.

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[34] A table containing numerical data for the solution of general MHD shocks may be regarded as a generalization of the familiar table for the solution of gasdynamic shocks (cf. NACA Ames Research Staff, Report 1135 (1953)). However, in MHD flows the presence of two additional independent parameters due to the magnetic field vector, in addition to the Mach number, renders a complete table covering full ranges of all characterizing parameters to be entirely unfeasible.

Appendix A

Using (5a) and (5b) to eliminate $\bar{u}$ and $\bar{B}$ in (2b, c, e, and f) and expressing them in terms of new variables we get

$$\frac{\bar{P}-1}{\gamma M_I^2} = \frac{\tau^2}{1+\tau^2}\left\{1 - \frac{1}{\bar{\rho}} + \frac{k_I[(\tau-\phi)^2(1+\tau s)^2-(\tau-s)^2(1+\tau\phi)^2]}{2M_I^2\tau^2(1+s^2)(\tau-\phi)^2}\right\} \tag{A1}$$

$$\bar{\rho}[1 + \frac{k_I(\phi-s)(1+\tau^2)(\tau-s)}{M_I^2\tau(1+s^2)(\tau-\phi)}] = \frac{\tau(1+\tau\varepsilon)}{\tau-\varepsilon} \tag{A2}$$

$$\bar{\rho} = \frac{\tau(\phi-\varepsilon)(\tau-s)}{s(\tau-\varepsilon)(\tau-\phi)} \tag{A3}$$

$$\bar{\rho}[\frac{M_I^2}{2} + \frac{1}{\gamma-1} + \frac{k_I s(1+\tau s)}{\tau(1+s^2)}] = \frac{M_I^2\tau^2(1+\varepsilon^2)}{2\bar{\rho}(\tau-\varepsilon)^2} + \frac{\bar{P}}{\gamma-1} + \frac{k_I(\tau-s)^2(\phi-\varepsilon)(1+\tau\phi)}{(\tau-\phi)^2(\tau-\varepsilon)(1+s^2)} \tag{A4}$$

These are four algebraic equations governing τ, ϕ, ε, $\bar{P}$, and $\bar{\rho}$, with known upstream parameters M_I, k_I, s and the given gas adiabatic exponent γ. $\varepsilon(\tau,\bar{\rho})$ and $\phi(\tau,\bar{\rho})$ may be readily solved from (A2) and (A3); they are given in (8b) and (8c). Elimination of ϕ in (A1) by the use of (8c) yields $\bar{P}(\tau,\bar{\rho})$ in (8d). Using (8b, c, and d) to eliminate ε, ϕ, and $\bar{P}$ in (A4), we get $f(\tau,\bar{\rho}) = 0$ in (8a). Elimination of ε in (5a) by the use of (8b) yields $\bar{u}^2(\tau,\bar{\rho})$ in (8e). Using (8c) to eliminate ϕ in (5b) yields $\bar{B}^2(\tau,\bar{\rho})$ in (8f).

Appendix B

Consider a quadratic and a cubic equation of x as follows:

$$ax^2 + bx + c = 0 \tag{B1}$$

$$dx^3 + ex^2 + fx + g = 0 \tag{B2}$$

Elimination of x from (B1) and (B2) yields the following equation relating the coefficients:

$$a[(ag-ce)^2 + (af-be)(cf-bg) + d[c^3d + bc(ag-ce) + (cf-bg)(b^2-2ac)] = 0 \tag{B3}$$

Since (8a) and (8b) are cubic and quadratic equations of $\bar{\rho}$ respectively, $\bar{\rho}$ can be easily eliminated by the use of (B3). The derivations are simpler if we use, instead of (8a) and (8b), the equivalent but more concise expressions in (11a) and (11b). Taking $x = \bar{\rho}/u_n^2$, we determine readily the expressions of the corresponding coefficients a, ..., g by comparing (B1) and (B2) with (11b) and (11a) respectively. Substitution of these into (B3) yields

$$b_n b_t^2 (u_n^2 - b_n^2) f^* = 0$$

where

$$f^* = f^*(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, \frac{b_n}{a}, \frac{b_t}{a}; \gamma) = (\varepsilon u_n + u_t)^3 b_n^3 [\sigma b_n^2 + (\gamma-1) u_n^2 b_t^2]$$

$$+ (\varepsilon u_n + u_t)^2 (u_n - \varepsilon u_t) \Big\{ b_t [\sigma b_n^2 + (\gamma-1) u_n^2 b_t^2][(\gamma-1) b_n^2 + (2-\gamma) u_n^2] +$$

$$+ u_n(u_n b_t - u_t b_n)\{[2\sigma+(\gamma+1)b^2]b_n^2 + (\gamma-2)u_n^2 b_t^2\}\}$$

$$+ (\varepsilon u_n+u_t)(u_n-\varepsilon u_t)^2\{u_n(u_n u_t - b_n b_t)\{[\sigma+\gamma b_t^2 + 2(\gamma+1)b_n^2](u_t b_n - 2u_n b_t)$$

$$+ (\gamma+1)u_n b_n^2 b_t\} + (u_n^2-b_n^2)\{\sigma\{b_n[\sigma+\gamma b_t^2-(\gamma+1)b_n^2] + \gamma u_n u_t b_t\}$$

$$+ u_n b_t[\gamma^2 u_t b_t^2 + (\gamma+1)(2-\gamma)b_n(2u_n b_t - u_t b_n)]\}\}$$

$$+ (\gamma+1)(u_n-\varepsilon u_t)^3\{u_n(u_n u_t - b_n b_t)^2(2u_n b_t - u_t b_n)$$

$$- (u_n^2-b_n^2)[(\gamma+1)u_n b_n^2(u_n b_t - u_t b_n)+(u_n u_t - b_n b_t)(\sigma b_n+\gamma u_n u_t b_t)]\}$$

with $\sigma \equiv 2a^2 + (\gamma-1)u_n^2$. Since $b_n \neq 0$, $b_t \neq 0$, and $u_n^2 \neq b_n^2$, in general

$$f^* = 0 \ . \tag{B4}$$

It gives the solution for compressive MHD shocks, and it is a polynomial of degree nine in u_n. Expressing (B4) in terms of M_I, k_I, s, and τ by the use of (10a), we get (17). (B4) is a convenient representation of the shock solution for one-dimensional unsteady flows.

Appendix C. Normal incidence ($u_t = 0$)

At $u_t = 0$, ε, ϕ, and $\bar{u}^2$ in (11b, c, and e) simplify considerably as shown in Section II, whereas $\bar{\rho}$, $\bar{P}$, and $\bar{B}^2$ in (11a, d, and f) remain unchanged. (B4) also reduces, after setting $u_t = 0$ and $u_n = u$ in it, to

$$u^3 I = 0 \tag{C1}$$

where

$$\begin{aligned} I = {} & \varepsilon^3 b_n^3 [2a^2 b_n^2 + (\gamma-1)u^2 b^2] \\ & + \varepsilon^2 b_t \Big\{ [2a^2 b_n^2 + (\gamma-1)u^2 b^2][(4-\gamma)u^2 + (\gamma-1)b_n^2] + (\gamma+1)b_n^2 b^2 \\ & - \gamma u^2 b_t^2 \Big\} + \varepsilon b_n \Big\{ [2a^2 + (\gamma-1)u^2] \big\{ [2a^2 + (\gamma-1)u^2 - (\gamma+1)b_n^2](u^2 - b_n^2) \\ & + b_t^2 [(2+\gamma)u^2 - \gamma b_n^2] \big\} + u^2 b_t^2 \big\{ 2\gamma b_t^2 + (\gamma+1)[(2\gamma-1)b_n^2 + 2(2-\gamma)u^2] \big\} \Big\} \\ & + 2(\gamma+1) b_n^2 b_t^2 [u^2 b_t^2 - (u^2 - a^2)(u^2 - b_n^2)] \ . \end{aligned}$$

Thus $I = 0$ is a cubic equation of u^2.

Appendix D

(a) Aligned flow

$\vec{B} = \text{const.} \times \rho\vec{u}$ in aligned flows, and hence $\psi_I = 0$, $\psi_F = \theta$, and $\bar{B} = \bar{\rho}\bar{u}$. Ahead of the shock we have

$$\frac{b_n^2}{u_n^2} = \frac{b_t^2}{u_t^2} = \frac{b^2}{u^2} = \frac{k_I}{M_I^2} = \kappa \tag{D1}$$

where the constant, κ, is defined. (D1) can be used to eliminate b_n and b_t in all equations. The original four independent parameters $(u_n/a,\ u_t/a,\ b_n/a,\ b_t/a)$ corresponding to $(M_I,\ k_I,$

s, τ) then reduce, in this special case, to three independent parameters (u_n/a, u_t/a, κ) corresponding to (M_I, k_I, τ). (11a) becomes

$$\kappa\bar{\rho}(\bar{\rho}-1)u_t^2 + (1-\kappa\bar{\rho})\left\{2\bar{\rho}a^2(1-\kappa\bar{\rho})-[\gamma+1-(\gamma-1)\bar{\rho}](u_n^2-\kappa\bar{\rho}u^2)\right\} = 0 \, . \tag{D2}$$

(11b)-(11f) simplify to

$$\begin{aligned} \varepsilon &= \phi = \frac{u_n u_t(\bar{\rho}-1)}{u_n^2+\bar{\rho}u_t^2 - \kappa\bar{\rho}u^2} \quad , \\ \bar{P} &= 1 + \frac{\gamma u_n^2(\bar{\rho}-1)}{\bar{\rho}a^2}\left\{1 - \frac{\kappa\bar{\rho}u_t^2[1+(1-2\kappa)\bar{\rho}]}{2u_n^2(1-\kappa\bar{\rho})^2}\right\} , \\ \bar{u}^2 &= \frac{\bar{B}^2}{\bar{\rho}^2} = \frac{1}{u^2}\left[\frac{u_n^2}{\bar{\rho}^2} + u_t^2\left(\frac{1-\kappa}{1-\kappa\rho}\right)^2\right] \, . \end{aligned} \tag{D3}$$

Also (B4) simplifies to

$$f^*(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, \kappa; \gamma) = \kappa^{\frac{1}{2}}u_n^3[2a^2+(\gamma-1)u^2]A^* = 0 \tag{D4}$$

where

$$\begin{aligned} A^* = {} & \kappa u_n(\varepsilon u_n+u_t)^2\left\{\kappa u_n(\varepsilon u_n+u_t) + u_t[2-\gamma+(\gamma-1)\kappa](u_n-\varepsilon u_t)\right\} \\ & + (1-\kappa)(u_n-\varepsilon u_t)^2\left\{\left\{2a^2+(\gamma-1)u_n^2+\kappa[\gamma u_t^2-(\gamma+1)u_n^2]\right\}(\varepsilon u_n+u_t)\right. \\ & \left. - (\gamma+1)(1-\kappa)u_n u_t(u_n-\varepsilon u_t)\right\} . \end{aligned}$$

$A^* = 0$ gives the shock solution, and it corresponds to $A = o$ in (18).

(b) Very weak magnetic field

Here $b_n/a \ll 1$ and $b_t/a \ll 1$, whereas b_n/b_t is finite. We get from (11a)-(11f)

$$\bar{\rho} = \frac{(\gamma+1)u_n^2}{2a^2+(\gamma-1)u_n^2} \,, \quad \varepsilon = \frac{u_n u_t(\bar{\rho}-1)}{u_n^2+\bar{\rho}u_t^2} \,, \quad \phi = \frac{\bar{\rho}u_n-u_t(b_n/b_t)}{\bar{\rho}u_t+u_n(b_n/b_t)}$$

$$\bar{P} = 1+\frac{\gamma u_n^2(\bar{\rho}-1)}{\bar{\rho}a^2} \,, \quad \bar{u}^2 = \frac{u_n^2+\bar{\rho}^2u_t^2}{\bar{\rho}^2u^2} \quad , \quad \bar{B}^2 = 1 + \frac{\bar{\rho}^2-1}{1+b_n^2/b_t^2} \; . \tag{D5}$$

Thus in this limit $\bar{\rho}$, ε, $\bar{P}$, and $\bar{u}^2$ are unaffected by the presence of the very weak magnetic field; they are just the well-known result for a gasdynamic shock. Conversely, the magnetic field behind the shock being characterized by ϕ, and $\bar{B}^2$, depends on the fluid properties and the shock strength parameter $\bar{\rho}$. Since $\bar{\rho} \geq 1$ for real shocks satisfying the entropy condition, we have $\bar{B}^2 \geq 1$ always in this limit.

(B4) becomes

$$f^*(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, \frac{b_n}{a} \ll 1, \frac{b_t}{a} \ll 1; \gamma) \tag{D6}$$

$$= u_n^2(u_n-\varepsilon u_t)^2\Big\{b_n[2a^2+(\gamma-1)u_n^2] + u_t[u_t b_n-(2-\gamma)u_n b_t]\Big\}W^* = 0$$

where

$$W^* = [2a^2+(\gamma-1)u_n^2(\varepsilon u_n+u_t)-(\gamma+1)u_n u_t(u_n-\varepsilon u_t) \; .$$

ε does not enter in the curly bracket of (D6); hence the shock solution is given by $W^* = 0$ which corresponds to $W = 0$ in (19).

(c) Very strong magnetic field

Here $b_n/a \gg 1$ and $b_t/a \gg 1$, whereas b_n/b_t is finite. (11a) becomes

$$\bar{\rho}^2 b_n^2 \left\{ 2\bar{\rho} a^2 b_n^2 - u_n^2 b^2 [\gamma+1-(\gamma-1)\bar{\rho}] \right\} = 0 . \tag{D7}$$

Apparently $\bar{\rho} = 0$ is not a physically realistic solution. Then we get from (D7) for $\bar{\rho} \neq 0$, together with (11b)-(11f),

$$\bar{\rho} = \frac{(\gamma+1)u_n^2}{2a^2/(1+b_t^2/b_n^2) + (\gamma-1)u_n^2} , \quad \varepsilon = \frac{u_n(\bar{\rho}-1)(u_t - u_n(b_t/b_n))}{u_n^2 + \bar{\rho}u_t^2 - u_n u_t(\bar{\rho}-1)(b_t/b_n)} ,$$

$$\phi = \frac{u_n(b_t/b_n) - u_t}{u_n + u_t(b_t/b_n)} , \quad \bar{P} = 1 + \frac{\gamma u_n^2(\bar{\rho}-1)}{\bar{\rho} a^2} \left(1 + \frac{b_t^2}{b_n^2}\right) \tag{D8}$$

$$\bar{u}^2 = \frac{1}{\bar{\rho}^2 u^2} \left\{ u_n^2 + \left[\bar{\rho} u_t - (\bar{\rho}-1) u_n \frac{b_t}{b_n}\right]^2 \right\} , \quad \bar{B}^2 = 1 .$$

Thus we have $\phi = s$ and $\bar{B}^2 = 1$ which means $\vec{B}_F = \vec{B}_I$. The very strong magnetic field remains unchanged across the shock while the fluid variables $\bar{\rho}$, ε, $\bar{P}$, and $\bar{u}^2$ are affected by the presence of the magnetic field only through its orientation given by b_t/b_n.

(B4) becomes

$$f^*\left(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, \frac{b_n}{a} \gg 1, \frac{b_t}{a} \gg 1; \gamma\right) \tag{D9}$$

$$= b_n[b_n(\varepsilon u_n + u_t)^2 + \gamma b_t(\varepsilon u_n + u_t)(u_n - \varepsilon u_t) + (\gamma+1) b_n (u_n - \varepsilon u_t)^2] S^* = 0$$

where

$$S^* = b_n[2a^2b_n^2 + (\gamma-1)u_n^2b^2](\varepsilon u_n+u_t)$$
$$- \left\{b_t[2a^2b_n^2 + (\gamma-1)u_n^2b^2] + (\gamma+1)u_nb^2(u_tb_n-u_nb_t)\right\}(u_n-\varepsilon u_t).$$

The term inside the square bracket in (D9) corresponds to $\bar{\rho} = 0$, so it is not a physical solution. Shock solutions are given by $S^* = 0$ which corresponds to $S = 0$ in (20).

It is of interest to observe from (D8) that $\bar{\rho}$ and $\bar{P}$ for this very strong magnetic field case can be obtained simply by replacing a^2 in their corresponding gasdynamic solution with $a^2/(1 + b_t^2/b_n^2)$.

(d) Zero flow deflection angle

$\varepsilon = 0$ in this case, and (11b) gives

$$u_n(\bar{\rho}-1)[u_t(u_n^2-\bar{\rho}b_n^2) + \bar{\rho}u_nb_nb_t] = 0 \quad . \tag{D10}$$

Let us first consider the case of $\bar{\rho} = 1$. (11a) becomes

$$2(u_n^2-b_n^2)Z_1^* = 0 \tag{D11}$$

where

$$Z_1^* = u_n^2b_t^2 - (u_n^2-a^2)(u_n^2-b_n^2) \quad .$$

However, ε is not necessarily zero if both $\bar{\rho} = 1$ and $u_n^2 = b_n^2$, because the denominator of (11b) becomes zero also (see IIB-i-g and Appendix D-g for further discussions). We consider here only $u_n^2 \neq b_n^2$. Then $Z_1^* = 0$ and (11c)-(11f) give $\phi = s$, $\bar{P} = \bar{u}^2 = \bar{B}^2 = 1$. These, together with $\varepsilon = 0$, show that the shocks are of zero strength across which the state of gas does not change at all. $Z_1^* = 0$ is the familiar equation for the characteristics, or the small disturbance wave speeds (i.e., the limiting solution of shocks of very weak strength). Two roots of u_n in $Z_1^* = 0$ describe fast waves whereas the other two roots describe slow waves in general. These are the only four solutions describing shocks of zero strength.

Next we consider the case that the square bracket in (D10) vanishes. This gives

$$\bar{\rho} = \frac{u_n^2 u_t}{b_n(u_t b_n - u_n b_t)} \tag{D12}$$

which describes shocks of finite strength. Since $\bar{\rho} \geq 1$ for a physical shock, (D12) yields the necessary condition that

$$u_n(u_n u_t = b_n b_t) \geq u_t b_n^2 \tag{D13}$$

for a physical shock. The corresponding expressions of ϕ, $\bar{P}$, $\bar{u}^2$, and $\bar{B}^2$ can be easily obtained by substituting $\bar{\rho}$ given by (D12) into (11c)-(11f).

Setting $\varepsilon = 0$ in (B4) yields

$$f^*(0, \frac{u_n}{a}; \frac{u_t}{a}, \frac{b_n}{a}, \frac{b_t}{a}; \gamma) = 2Z_1^* Z_2^* = 0 \tag{D14}$$

where

$$Z_1^* = (u_n b_t - u_t b_n)[u_t^2(u_n^2 - b_n^2) - \gamma u_n u_t b_n b_t - (\gamma+1)u_n^2 b_n^2]$$
$$- u_n^2 u_t[2a^2 b_n + (\gamma-1)u_n(u_n b_n + u_t b_t)] \ .$$

$Z_1^* = 0$ corresponds to $Z_1 = 0$ in (21). $Z_2^* = 0$ corresponds to $Z_1 =$ in (21). In the limit of very weak magnetic field ($b_n/a \ll 1$, $b_t/a \ll 1$, and b_n/b_t finite), $Z_1^* = 0$ gives $u_n^2 = a^2$ and it corresponds to sound waves in gasdynamics, while $Z_1^* =$ gives $u_t = 0$ (or $\tau \rightarrow \infty$); hence $\varepsilon = 0$ which is the solution for a normal shock wave of finite strength.

(e) Normal upstream magnetic field ($B_{I,t} = 0$)

Consider this first as the limit of $B_{I,t} \rightarrow 0$ so that $\vec{e}_t$ and $\vec{e}_z$ are well defined. Setting $b_t \rightarrow 0$ in (11a) gives

$$(u_n^2-\bar{\rho}b_n^2)^2 \quad 2\bar{\rho}a^2-u_n^2[\gamma+1-(\gamma-1)\bar{\rho}] \quad = \quad 0 \tag{D15}$$

Let us consider $u_n^2 \neq \rho b_n^2$ first. The vanishing of the curly bracket in (D15) yields the same solution of $\bar{\rho}$ as that for a gasdynamic shock given in (D5). Setting $b_t = 0$ in (11b)-(11f), we get $\phi = -u_t/u_n = -\tau^{-1} = s$, $\bar{B}^2 = 1$, whereas ε, $\bar{P}$, and $\bar{u}^2$ are precisely the gasdynamic solutions given in (D5). Thus we have for this case a pure gasdynamic shock with $\vec{B}_F = \vec{B}_I$. Obviously, u_t remains unchanged across this shock and a normal shock exists if $u_t = 0$.

Next we consider $u_n^2 = \bar{\rho}b_n^2$ which is a double root here. Straightforward substitution of $b_t = 0$ into (11b)-(11f) gives rise to a set of indeterminant equations. A precise examination of the limiting process is necessary. It follows from (11a) that as $b_t \rightarrow 0$ and $u_n^2-\bar{\rho}b_n^2 \rightarrow 0$, $b_t^2(u_n^2-\bar{\rho}b_n^2)^{-2}$ must remain finite. To approach the limit properly it is convenient to introduce

$$b_t = \delta x \quad \text{and} \quad u_n^2 - \bar{\rho} b_n^2 = \delta y \tag{D16}$$

where x and y remain finite as the small parameter δ tends to zero. Denoting $\xi = (b_n b_t \bar{\rho}(\bar{\rho}-1))/(u_n^2 - \bar{\rho} b_n^2) = (x b_n \bar{\rho}(\bar{\rho}-1))/y$, we get from (11a) as $\delta \rightarrow 0$ (hence $b_n \rightarrow b$ and $\bar{\rho} \rightarrow u_n^2/b^2$),

$$\xi = \pm \left\{ (\bar{\rho}-1)[2(1-\frac{a^2}{b^2}) - (\gamma-1)(\bar{\rho}-1)] \right\}^{\frac{1}{2}} \tag{D17}$$

which is finite. The sign in front of the radical in (D17) should be determined by the physical boundary condition in the problem. Using (D16) to eliminate b_t and u_n in (11b)-(11f) and then taking the limit of $\delta \rightarrow 0$, we get

$$\varepsilon = \frac{u_n[u_t(\bar{\rho}-1) + u_n\xi]}{u_n^2 + \bar{\rho} u_t^2 + u_n u_t \xi} \;, \quad \phi = \frac{u_n\xi - u_t}{u_n + u_t\xi}$$
$$\bar{P} = 1 + \frac{\gamma b^2}{a^2}(\bar{\rho}-1-\frac{\xi^2}{2}) \;, \quad \bar{u}^2 = \frac{u_n^2 + (\bar{\rho} u_t + u_n\xi)^2}{\bar{\rho}^2 u^2} \;, \quad \bar{B}^2 = 1+\xi^2, \tag{D18}$$

where $u_n^2 = \bar{\rho} b^2$. Since $u_n/u_t = \tau = -s^{-1}$ here, ξ can be solved from the equation for ϕ in (D18) as

$$\xi = \frac{\phi - s}{1+\phi s} = \tan(\psi_F - \psi_I) \quad \text{or} \quad \psi_F = \psi_I + \tan^{-1}\xi \;. \tag{D19}$$

Hence ξ represents the tangent of the deflection angle of the magnetic field across the shock. Its sign can be easily determined here as required by the boundary conditions. Since $B_I^2 = B_{I,n}^2 = B_{F,n}^2$, we have also from the equation for $\bar{B}^2$ in (D18)

$$B_{F,t}^2 = B_I^2(\bar{B}^2 - 1) = B_I^2\xi^2 \tag{D20}$$

which is equivalent to (D19). Thus a finite tangential magnetic field behind the shock appears, whereas it is absent ahead of the shock; this type of shock is called a complete switch-on shock[4,6,13] and obviously it belongs to fast shocks.

Since $\bar{\rho} > 1$ is required by the entropy increase condition for finite shocks, we have from $u_n^2 = \bar{\rho} b^2$ that

$$u_n^2 > b^2 \ . \tag{D21}$$

Also since $\xi^2 > 0$ is necessary, whereas $\bar{\rho} > 1$ and $\gamma > 1$ always, (D17) yields

$$b^2 > a^2 + \frac{\gamma - 1}{2} b^2(\bar{\rho} - 1) > a^2 \ . \tag{D22}$$

(D21) and (D22) are two necessary conditions for the existence of a complete switch-on shock. Evidently ξ is the most natural parameter to describe a switch-on shock. If $\xi = 0$, (D18) reduces to the solution describing a pure gasdynamic shock as discussed previously.

$\bar{u}_t^2 = (\bar{u}^2(1+\tau\varepsilon)^2)/(1+\varepsilon^2)$ by definition and we get, after eliminating ε by the use of ε in (D18),

$$u_{F,t} = \left(u_t + \frac{\xi u_n}{\bar{\rho}}\right)^2 \ . \tag{D23}$$

Hence, normal shock does not exist unless $\xi = 0$ and $u_t = 0$.

In terms of M_I, k_I, and s, $u_n^2 = \bar{\rho} b^2$ becomes

$$\bar{\rho} = \frac{M_I^2}{k_I(1+s^2)} \; . \tag{D24}$$

The conditions in (D21) and (D22) become

$$M_I^2 > k_I(1+s^2) \quad \text{and} \quad k_I > 1 \; , \tag{D25}$$

respectively. Using (D24) to eliminate $\bar{\rho}$ in (D17) yields

$$\xi^2 = [\frac{M_I^2}{k_I(1+s^2)} - 1]\left\{2(1 - \frac{1}{k_I}) - (\gamma-1)[\frac{M_I^2}{k_I(1+s^2)} - 1]\right\} \tag{D26}$$

Substitution of (D26) into (D20) yields $B^2_{F,t} = B^2_{F,t}(M_I, k_I, s; \gamma)$ in (24).

Setting $b_t = 0$ and $b_n = b$ in (B4) yields

$$f^*(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, \frac{b}{a}, 0; \gamma) = bN_1^* N_2^* = 0 \tag{D27}$$

where

$$N_1^* = N_1^*(\varepsilon, \frac{u_n}{a}) = [2a^2 + (\gamma-1)u_n^2](\varepsilon u_n + u_t) - (\gamma+1)u_n u_t (u_n - \varepsilon u_t)$$

$$N_2^* = N_2^*(\varepsilon, \frac{u_n}{a}) = [b^2(\varepsilon u_n + u_t) - u_n u_t (u_n - \varepsilon u_t)]^2$$
$$+ (u_n - \varepsilon u_t)^2 (u_n^2 - b^2)[2a^2 + (\gamma-1)u_n^2 - (\gamma+1)b^2].$$

$N_1^* = 0$ coincides with $W^* = 0$ in (D6) for a gasdynamic shock. It describes the case of $u_n^2 \neq \bar{\rho} b_n^2$ and corresponds to $N_1 = 0$ in (22). $N_2^* = 0$ describes the case of $u_n^2 = \bar{\rho} b_n^2$ for switch-on shocks. It can also be obtained readily by eliminating ξ in the equations for ε in (D18) with the aid of (D17) and setting $\bar{\rho} = u_n^2/b^2$. $N_2^* = 0$ corresponds to $N_2 = 0$ in (22).

For the special case of aligned flows that $u_t = 0$, ε, ϕ, and $\bar{u}^2$ simplify to

$$\varepsilon = \phi = \xi \quad \text{and} \quad \bar{u}^2 = \frac{1+\xi^2}{\bar{\rho}^2} \quad , \tag{D28}$$

whereas $\bar{P}$ and $\bar{B}^2$ remain unchanged as given in (D18). These agree with the result obtained by Friedrichs, and Bazer and Ericson.[6]

If $\vec{B}_I \times \vec{e}_n = 0$ strictly, there is no unique way of defining $\vec{e}_t$ and also $\vec{e}_z$. A degenerate two-parameter family of solutions exists. For the case of a pure gasdynamic shock ($N_1^* = 0$), the determination of $\vec{e}_t$ and $\vec{e}_z$ are entirely irrelevant. For the case of a complete switch-on shock ($N_2^* = 0$), the finite magnetic field component on the transverse plane (normal to $\vec{e}_n$) downstream, $\vec{B}_{F,T}$, may be directed at any angle on the boundary conditions. Then we have $\vec{e}_t \| \vec{B}_{F,T}$ ($= \vec{B}_{F,t}$), and $\vec{e}_z$ is also defined.

A switch-on shock differs from an Alfvén shock in that $\bar{\rho} \neq 1$ and $\bar{B}_T^2 \neq 1$. It is a compressive shock and entropy increases across it. However, $u_n^2 = \bar{\rho} b_n^2$ means simply $u_{F,n}^2 = b_{F,n}^2$. Thus a switch-on shock may be followed immediately by a non-compressive Alfvén shock across which $\vec{B}_{F,t}$ may further rotate through an arbitrary angle in the transverse plane, i.e., the (t,z) plane, with its strength unchanged while the flow velocity changes both its magnitude and direction accordingly.

(f) Tangential magnetic field ($B_{I,n} = 0$)

Setting $b_n = 0$, hence $b_t = b$ in (11a)-(11f) yields

$$2\bar{\rho}a^2 - u_n[\gamma+1-(\gamma-1)\bar{\rho}] + \bar{\rho}b^2[\gamma+(2-\gamma)\bar{\rho}] = 0$$

$$\varepsilon = \frac{u_n u_t(\bar{\rho}-1)}{u_n^2 + \bar{\rho}u_t^2} , \quad \emptyset = \tau(=s) , \quad \bar{P} = 1 + \frac{\gamma u_n^2(\bar{\rho}-1)}{a^2\bar{\rho}}\left[1 - \frac{b^2\bar{\rho}(\bar{\rho}+1)}{2u_n^2}\right] \quad \text{(D29)}$$

$$\bar{u}^2 = \frac{u_n^2 + \bar{\rho}u_t^2}{\bar{\rho}^2 u^2} , \qquad \bar{B}^2 = \bar{\rho}^2 .$$

The first of the above equations is a quadratic equation of $\bar{\rho}$ now. Since $\bar{u}_t^2 = (\bar{u}^2(1+\tau\varepsilon)^2)/(1+\varepsilon^2)$ by definition, we get, by the use of (D29), $\bar{u}_t^2 = 1$ always, as expected. Normal shocks exist at $u_t = 0$ for which (D29) yields $\varepsilon = 0$ and $\bar{u}^2 = \bar{u}_n^2 = \frac{1}{\bar{\rho}^2}$ with the other expressions unchanged.

Setting $b_n = 0$ in (B4) yields

$$f^*(\varepsilon, \frac{u_n}{a}; \frac{u_t}{a}, 0, \frac{b_t}{a}; \gamma) = (2-\gamma)u_n^3 b(u_n-\varepsilon u_t)T^* = 0 \qquad \text{(D30)}$$

where

$$T^* = (\gamma-2)u_n b^2(\varepsilon u_n+u_t)^2 - u_t[2a^2+(\gamma-1)u_n^2+\gamma b^2](u_n-\varepsilon u_t)(\varepsilon u_n+u_t)$$
$$+ (\gamma+1)u_n u_t^2(u_n-\varepsilon u_t)^2 .$$

$T^* = 0$ corresponds to $T = 0$ in (26).

(g) Non-characteristic solutions with $\bar{\rho} = 1$

It was shown in Appendix D-d that four solutions of τ at $\bar{\rho} = 1$ correspond to shocks of zero strength across which the state of gas does not change at all. They give the characteristic solutions and may be regarded as the limiting solutions of very weak shocks. However, $f(\tau,\bar{\rho}) = 0$ as given in (8a) is a polynomial of degree six in τ. For any given $\bar{\rho}$, six real and distinct roots of τ exist in general. The other two solutions of τ at $\bar{\rho} = 1$ are given by taking the limit of $\bar{\rho} \rightarrow 1$ and $u_n^2 \rightarrow b_n^2$ simultaneously, whereas the latter condition has, so far, been assumed to be not valid.

To obtain solutions for this case, it is convenient to define $\bar{\rho} = 1 + \delta x$, $u_n^2 = b_n^2 + \delta y$ where x and y remain finite as $\delta \rightarrow 0$. Substituting these into (11a) and taking the limit of $\delta \rightarrow 0$, we get

$$y = \frac{1}{2} b_n^2 x \ . \tag{D31}$$

Since $\delta x \geq 0$ is necessary according to the entropy condition, $u_n^2 - \bar{\rho} b_n^2 \rightarrow -\delta x(b_n^2/2) \leq 0$ in the limit. Thus only slow shocks are associated with this case. Similarly we get from (11b)-(11f), with the aid of (D31), the following solutions in the limit:

$$\varepsilon = [\frac{u_t}{u_n} - \frac{1}{2} \frac{b_n}{b_t} (1 + \frac{u_t^2}{u_n^2})]^{-1} \tag{D32}$$

$$\phi = \frac{u_n b_t + u_t b_n}{u_t b_t - u_n b_n} \tag{D33}$$

$$\bar{u}^2 = \frac{1}{u^2} [u_n^2 + (u_t - 2u_n \frac{b_t}{b_n})^2 \tag{D34}$$

$$\bar{P} = \bar{B}^2 = 1 \ . \tag{D35}$$

Unless $b_t = 0$ which corresponds to a special case, $\varepsilon \neq$, $\phi \neq s$, and $\bar{u}^2 \neq 1$ in general. Hence $\vec{u}_F \neq \vec{u}_I$ and $\vec{B}_F \neq \vec{B}_I$. The shock is of finite strength even though $\bar{\rho} = \bar{P} = \bar{B}^2 = 1$. It is not appropriate to consider $\bar{\rho}$ as a shock strength parameter here for it would easily lead to the misleading conception that $\bar{\rho} \longrightarrow 1$ means shocks of zero strength.

a^2 does not appear in (D31)-(D35). Therefore the inertial energy is irrelevant. The change of state across this type of shock is essentially kinematic in nature. ϕ in (D33) means

$$\psi_F = 2\beta - \pi - \psi_I \ . \tag{D36}$$

This, together with $\bar{B}^2 = 1$, shows that across this shock the change of magnetic field occurs only by reversing its tangential component, i.e., $B_{F,t} = -B_{I,t}$. It has precisely the same effect as the change across an Alfvén (or transverse) shock with the tangential magnetic field being rotated through an angle, π, in the transverse plane, together with the accompanying change of the flow velocity accordingly.

Expressing $u_n^2 = b_n^2$ and (D32) in terms of τ, M_I, k_I, and s, we get (28a) and (28b).

It is also of interest to see that for aligned flows, (D32)-(D35) reduce to

$$\varepsilon = \phi = \frac{2u_n u_t}{u_t^2 - u_n^2} \tag{D37}$$

$$\bar{P} = \bar{u}^2 = \bar{B}^2 = 1 \tag{D38}$$

simply; they are called improper slow shocks in ref. 6. The only changes across the shock in this case are the reversal of the tangential components of both flow velocity and magnetic field.

Figure Captions

Fig. 1: Velocity and magnetic field vectors on both sides of a general oblique MHD shock.

Fig. 2: The shock adiabatics on (β, θ) plane for $\gamma = 5/3$, $M_I = 3.0$, and $\psi_I = 30^\circ$. The values on the upper right corner of each graph signify those of $(M_I, k_I, \psi_I; \gamma)$ respectively. ω_+ and ω_- denote the fast and slow characteristics (or waves) respectively. r denotes the non-characteristic solution with $\bar{\rho} = 1$.

Fig. 3: Fig. (3a) shows the positions of the terminals of $-\bar{u}_I$ relative to the characteristic locus at $k_I = 0.5$ and 2.0, respectively, for each of the cases shown in Figs. (3b-n). All speeds are normalized with respect to a_I there. Figs. (3b-n) illustrate the shock adiabatics on the (β, θ) plane for $\gamma = 5/3$, $k_I = 0.5$ and 2.0, $\psi_I = 5^\circ$ and 0°; the values on the upper right corner of each of them signify those of $(M_I, k_I, \psi_I; \gamma)$ respectively. ω_+ and ω_- denote the fast and slow characteristics (or waves) respectively. r denotes the non-characteristic solution with $\bar{\rho} = 1$.

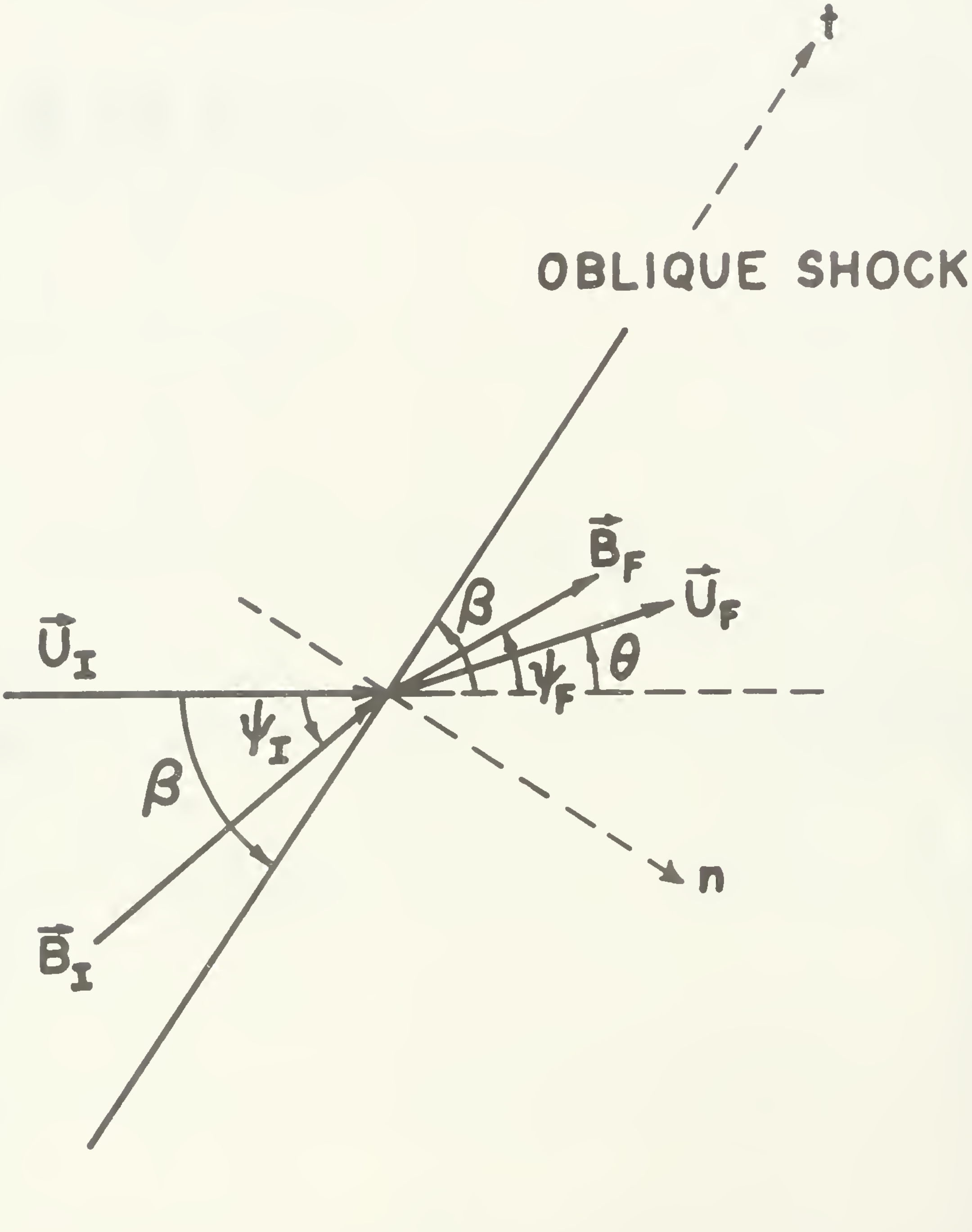
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Fig. 1

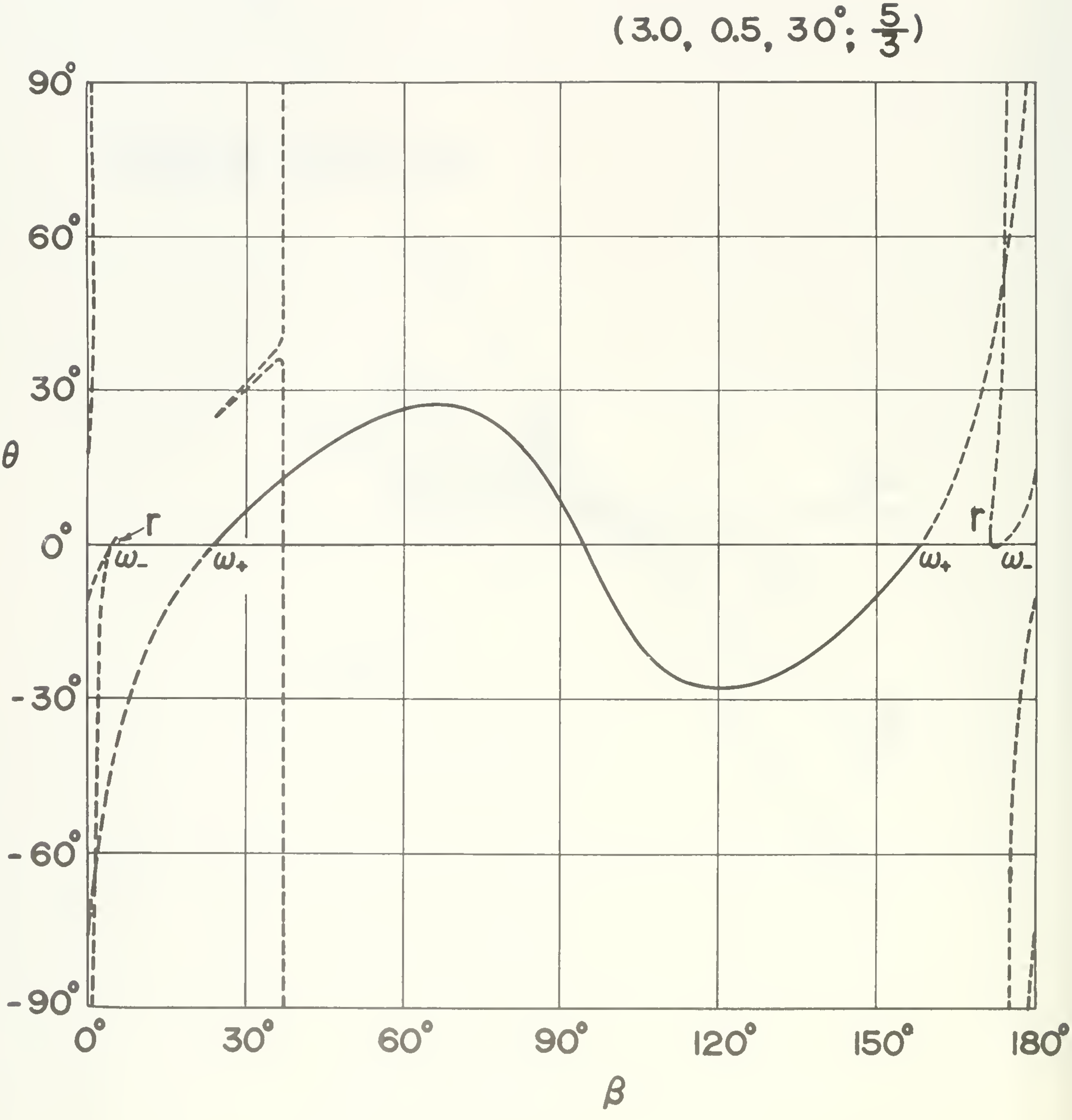

Fig. 2a

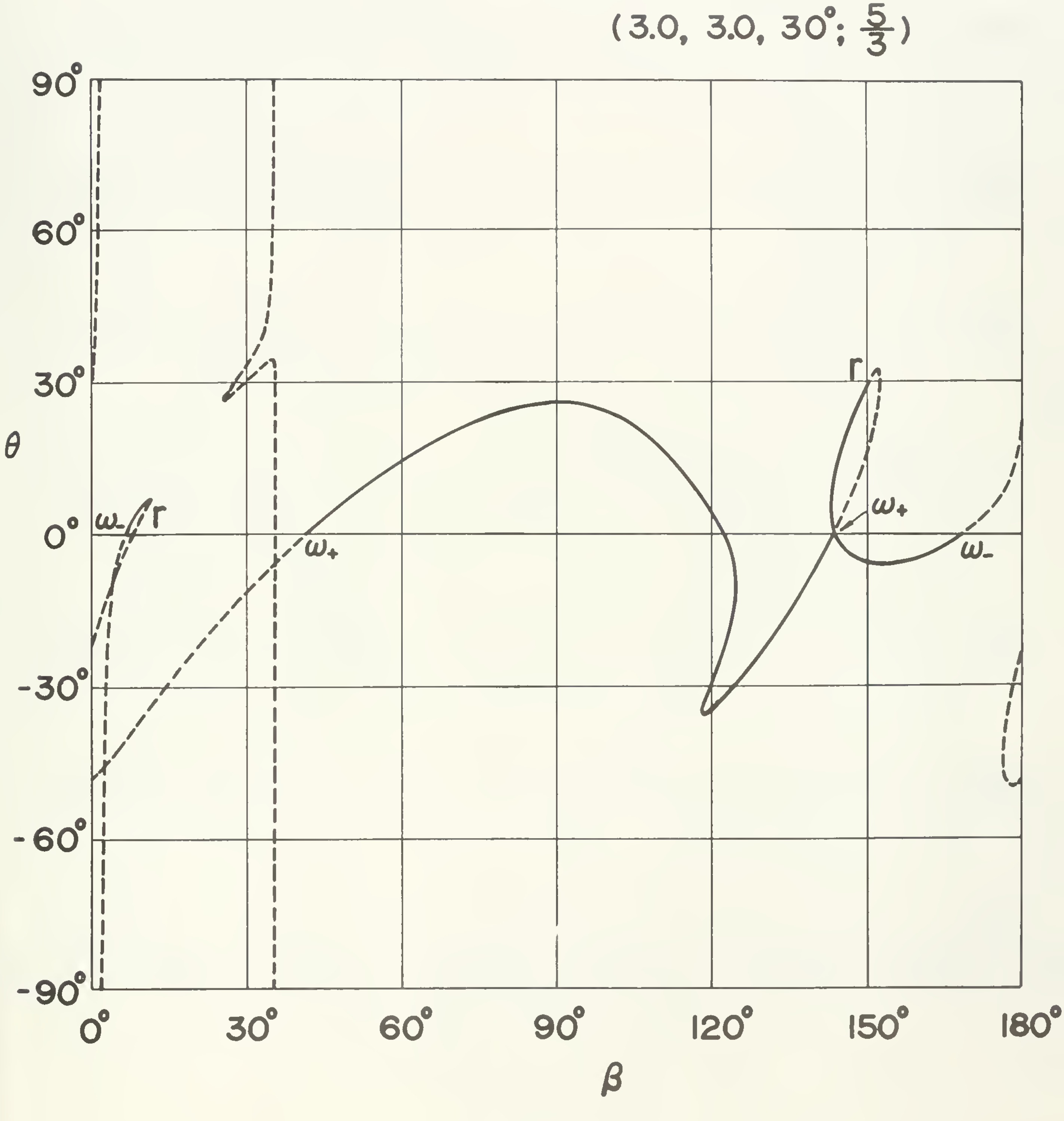

Fig. 2b

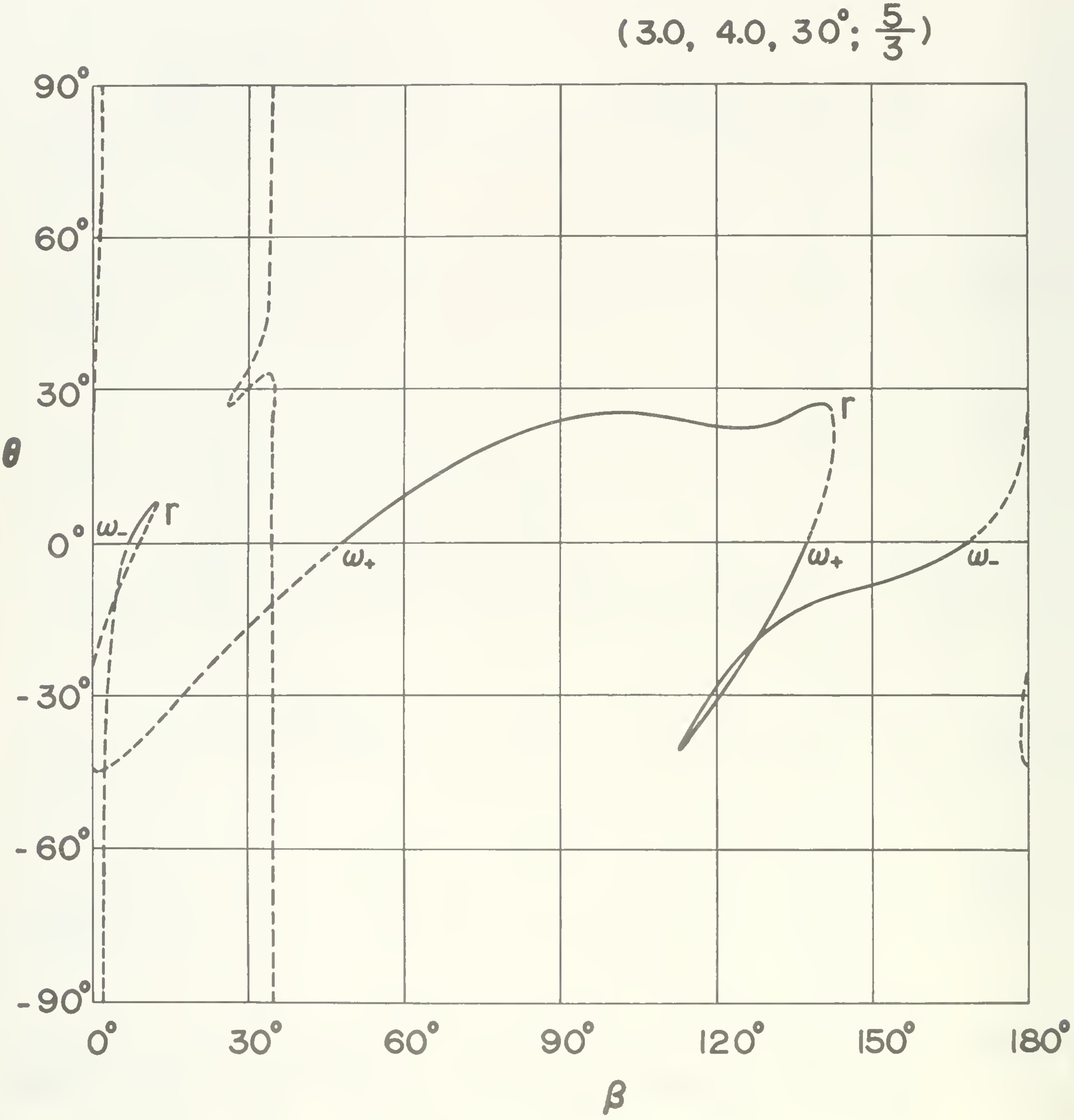

Fig. 23

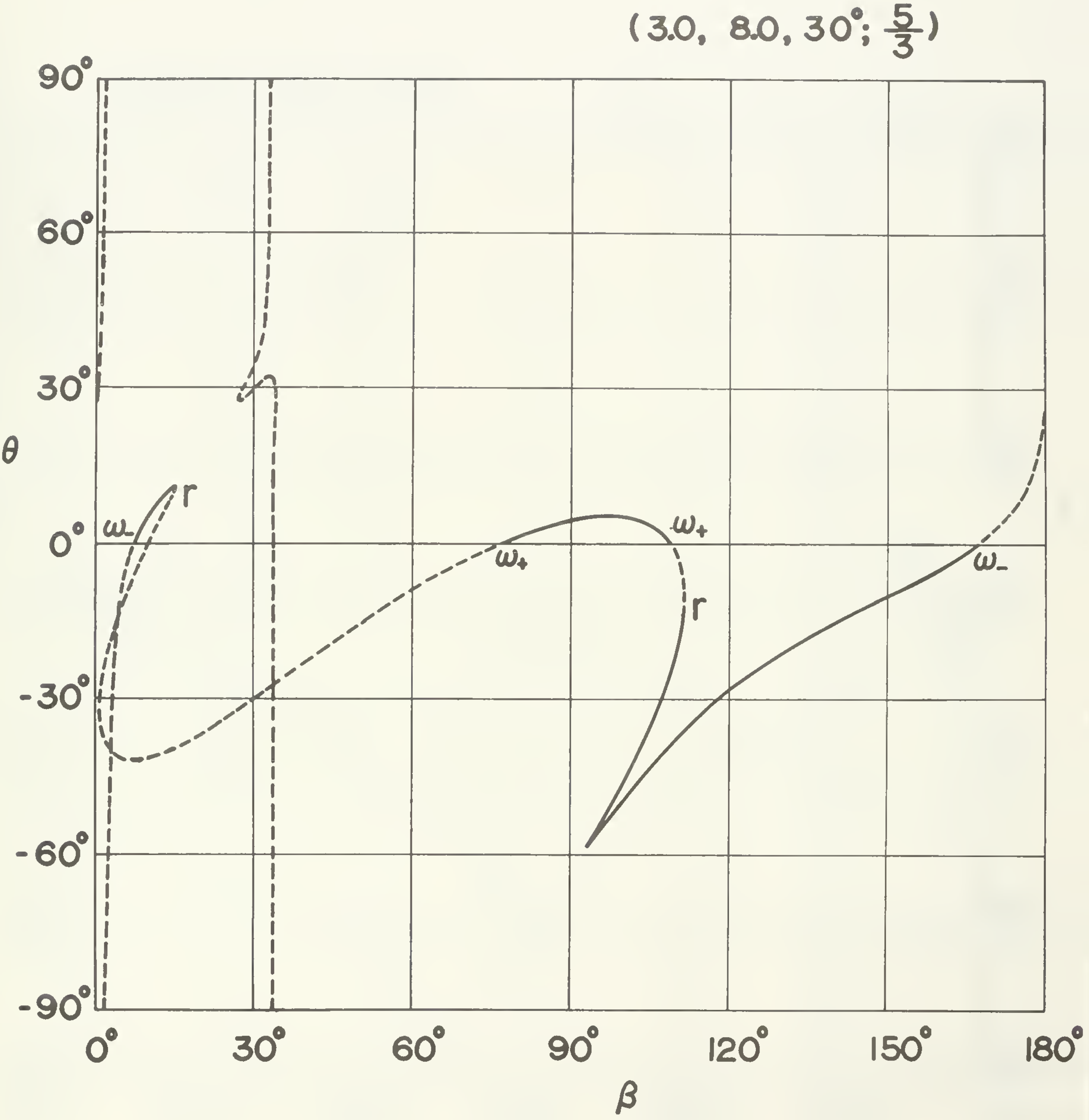

Fig. 2d

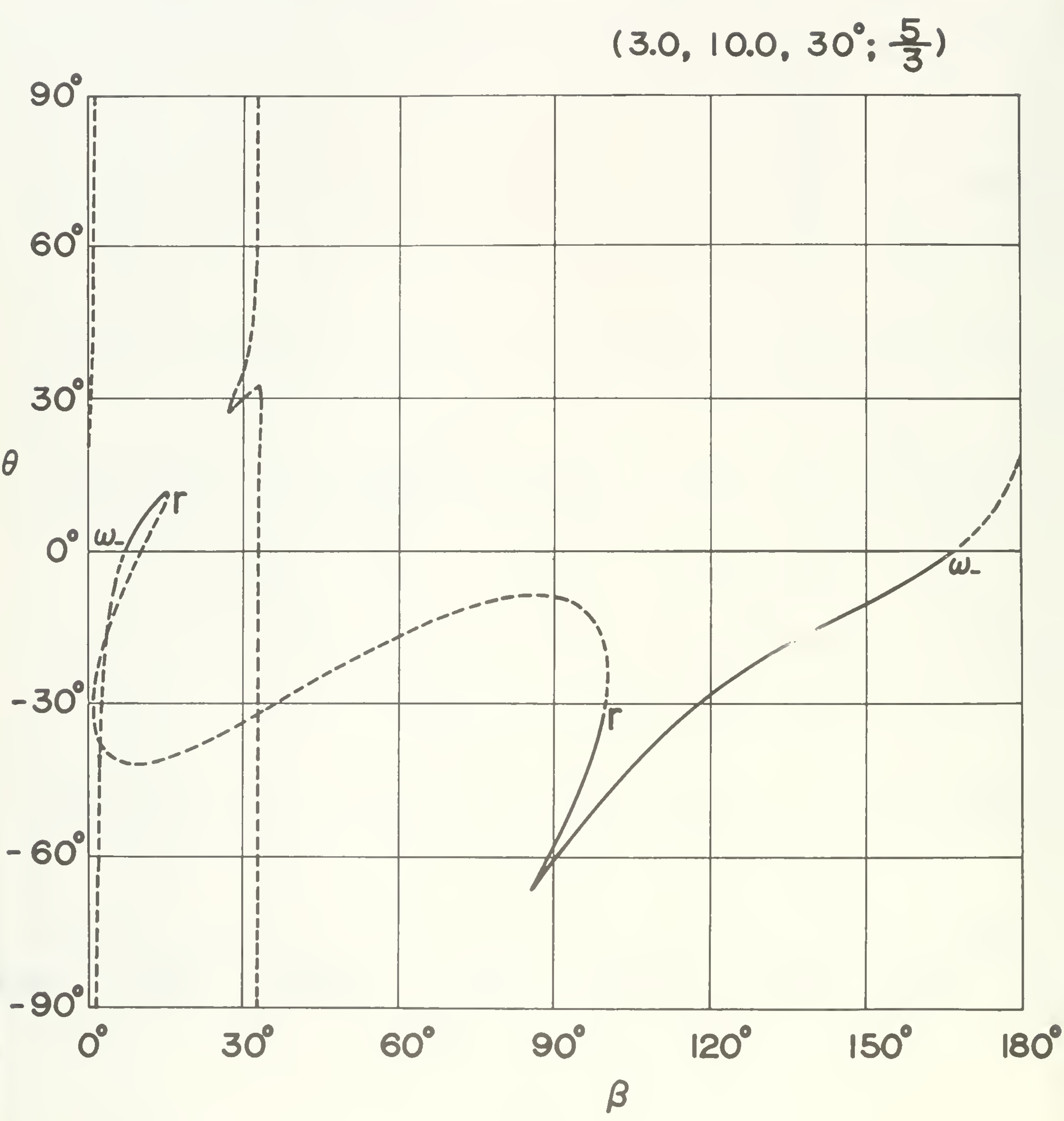

Fig. 2e

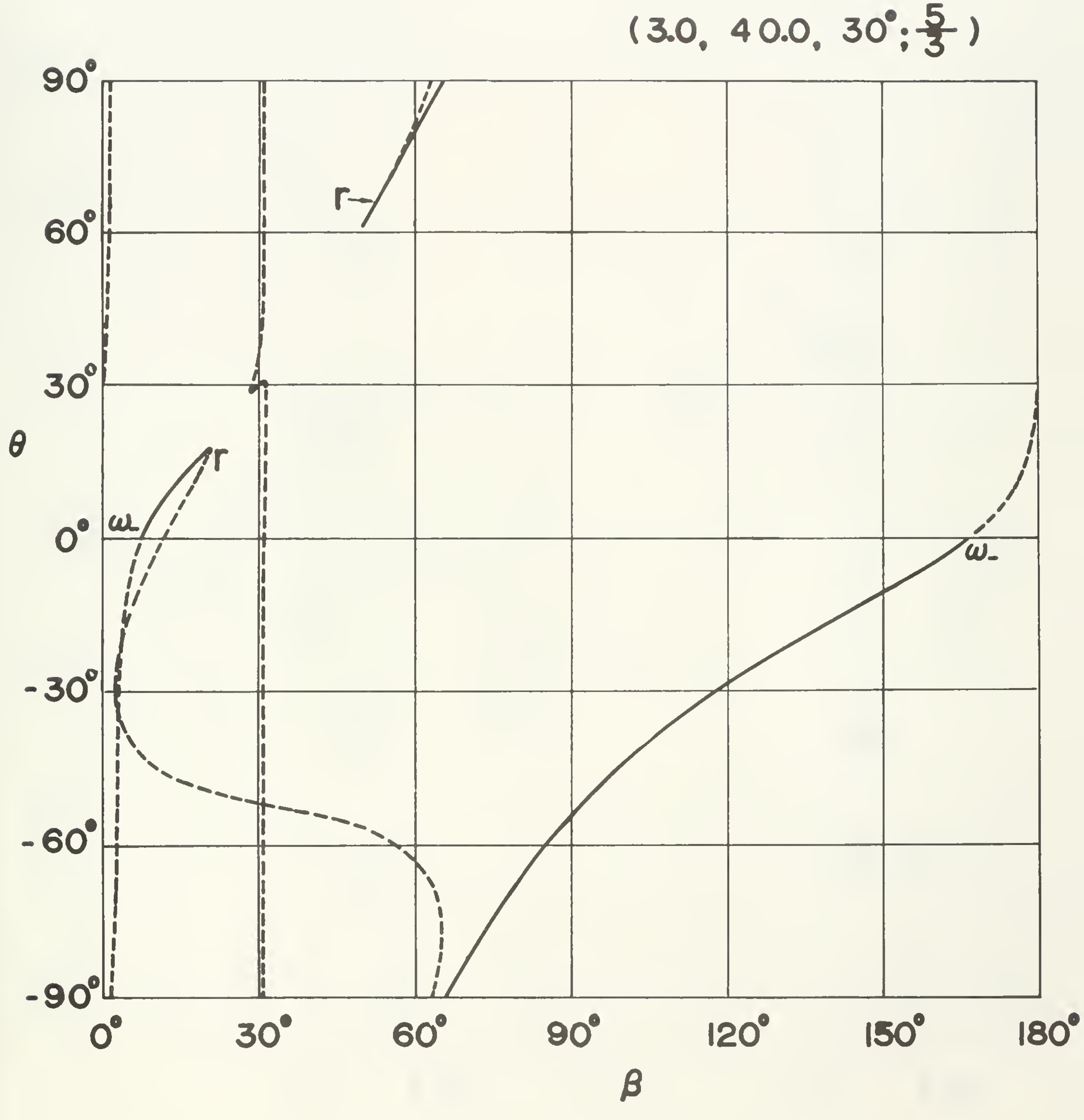

Fig. 2f

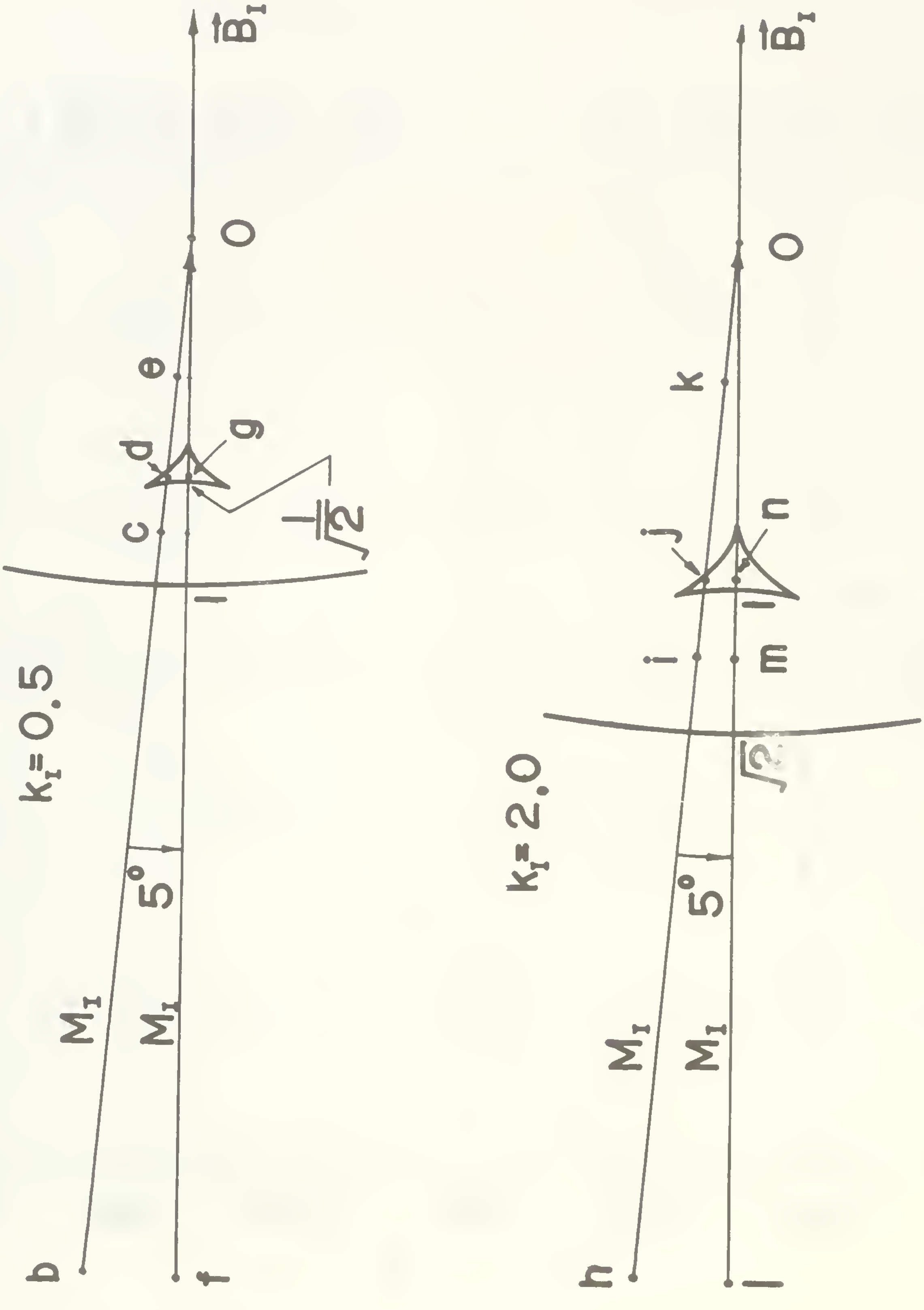
$k_I = 0.5$
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Fig. 3a

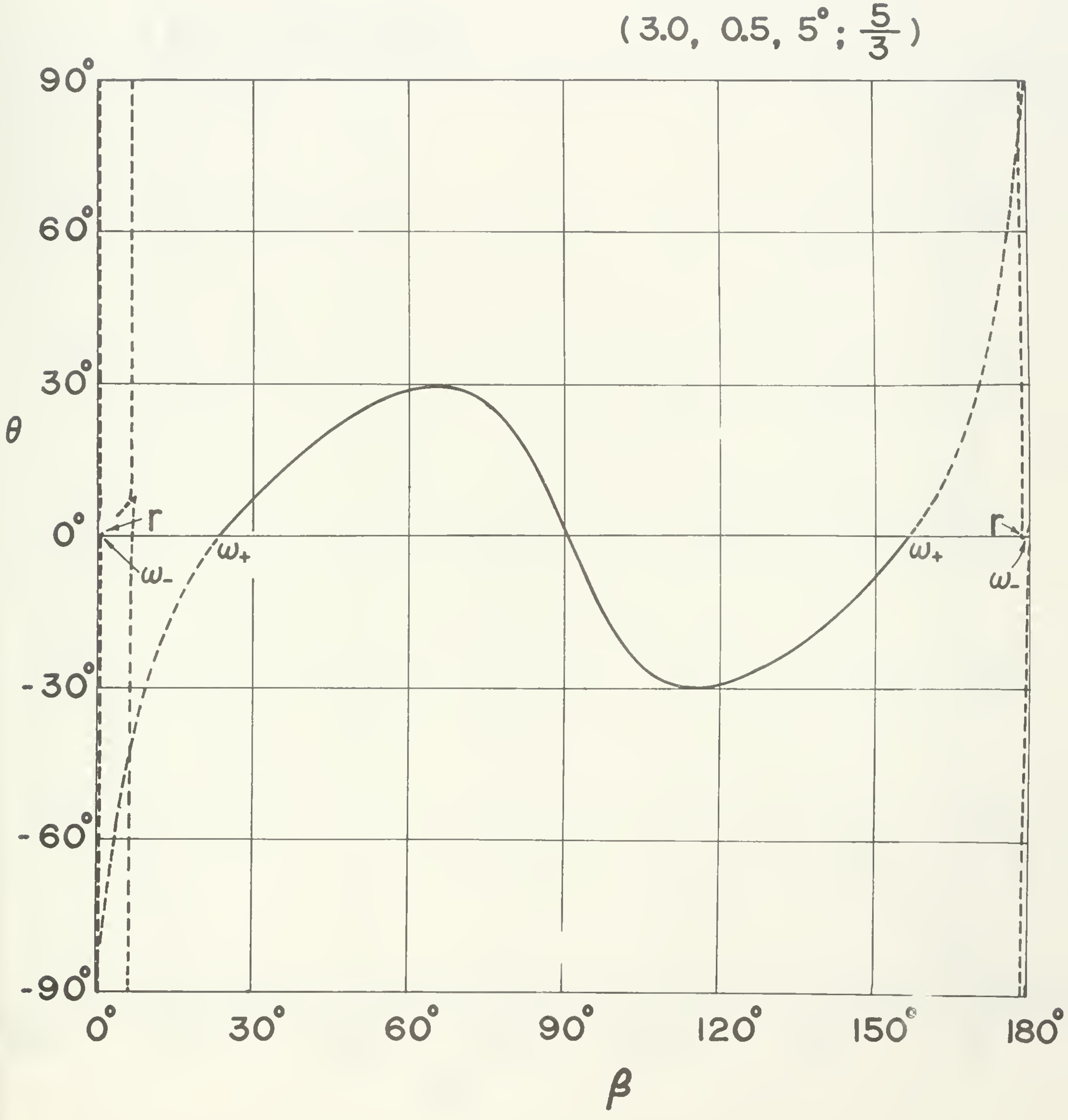


Fig. 3b

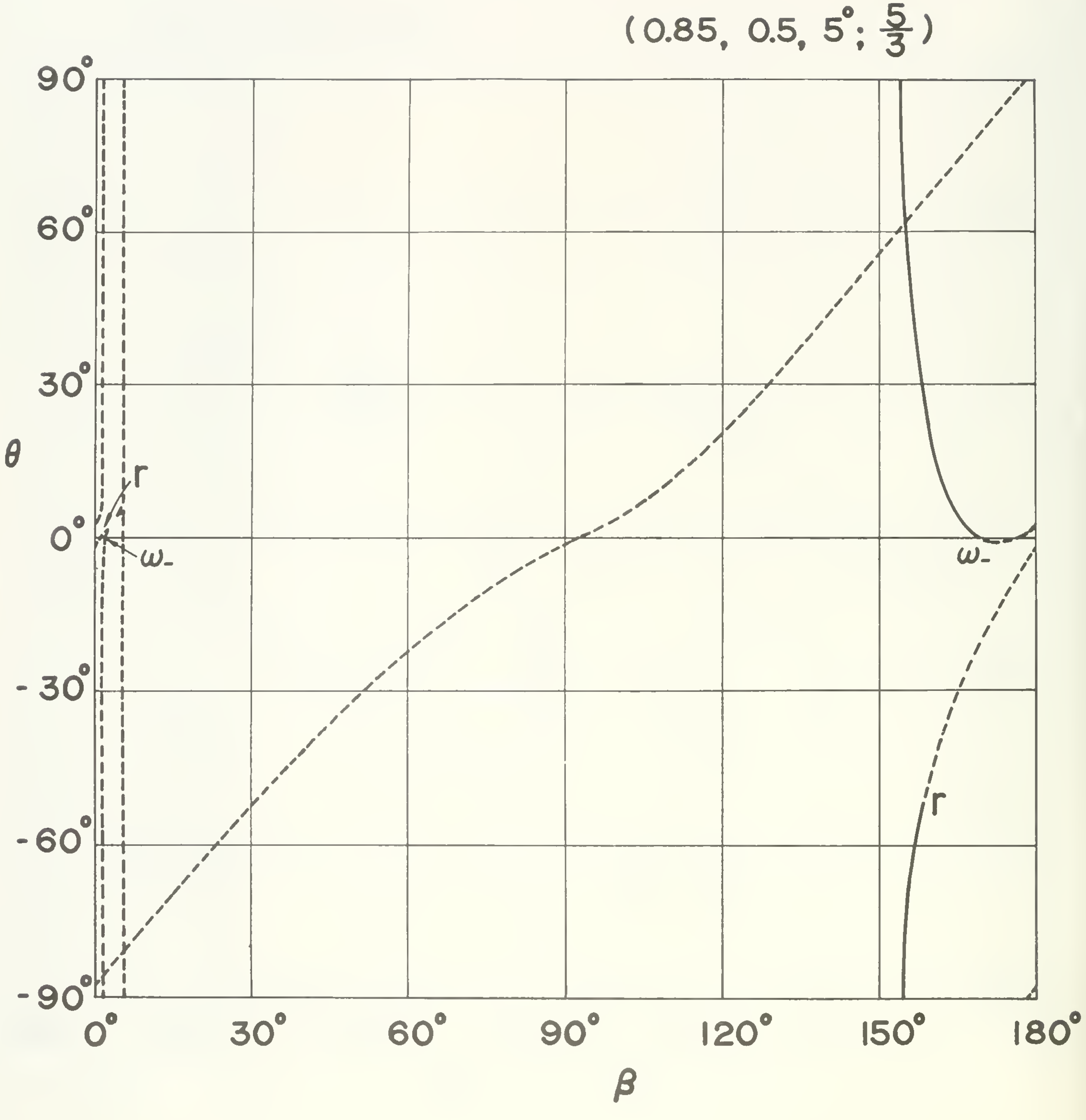


Fig. 3c

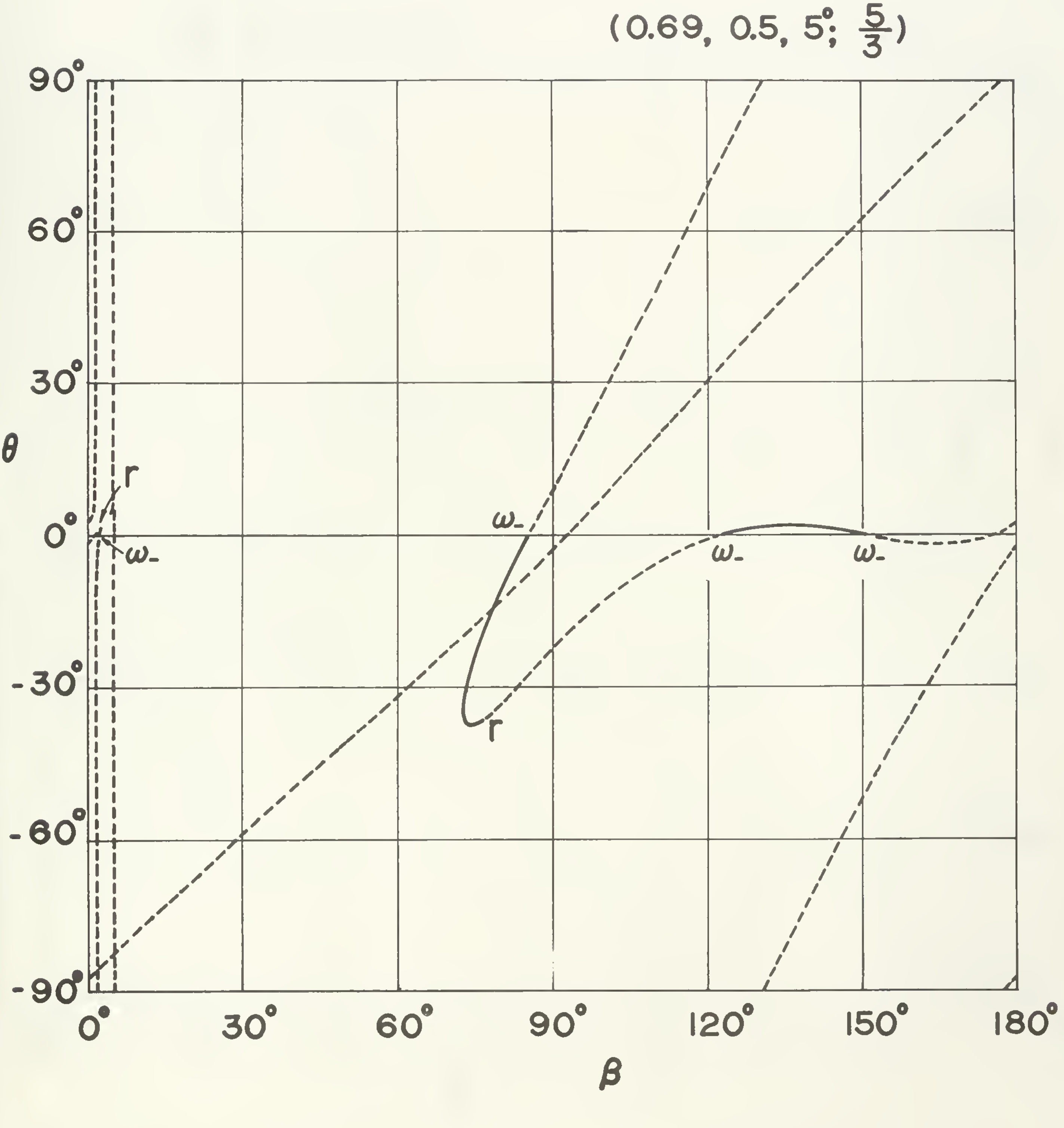


Fig. 3d

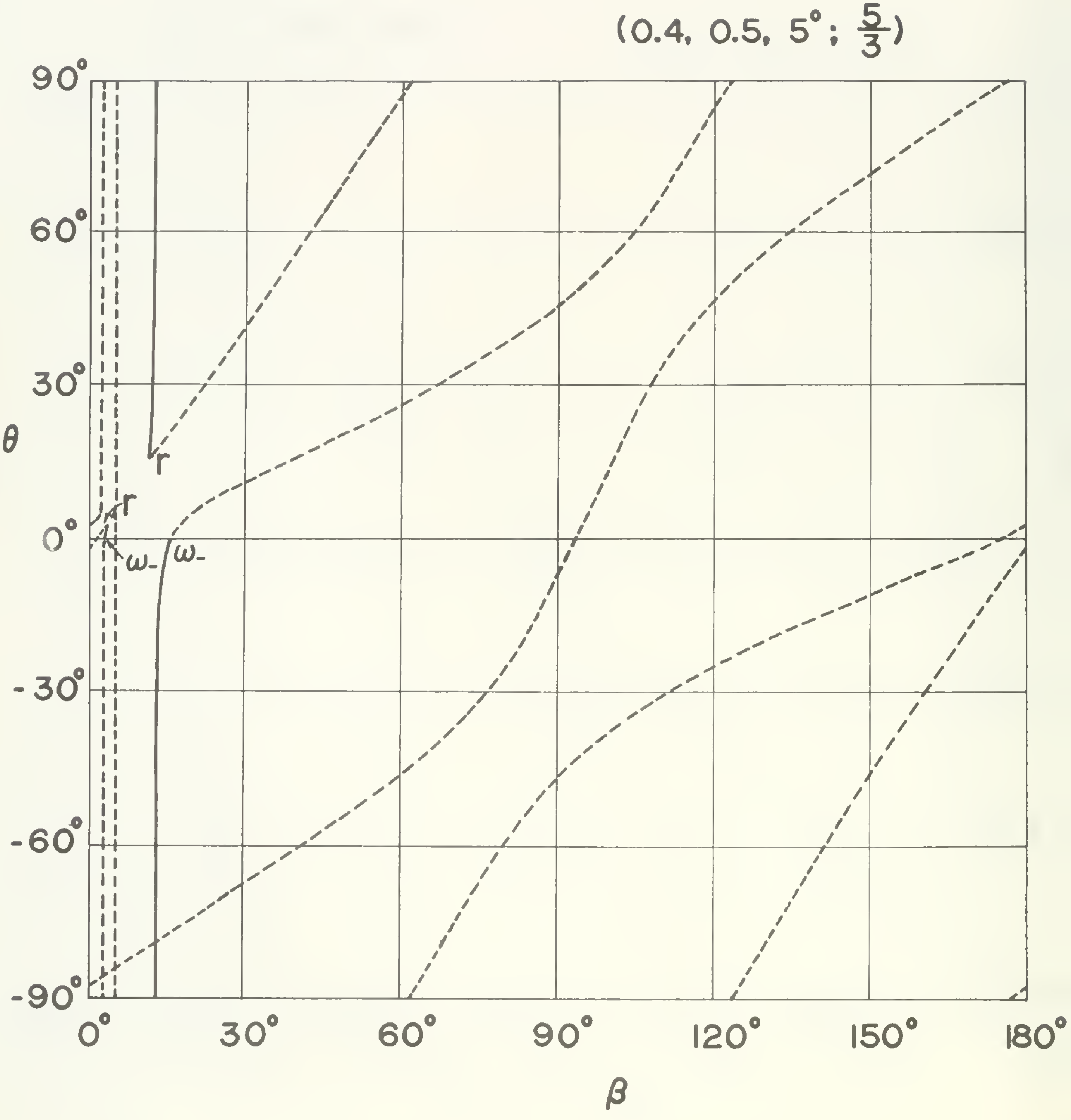


Fig. 3e

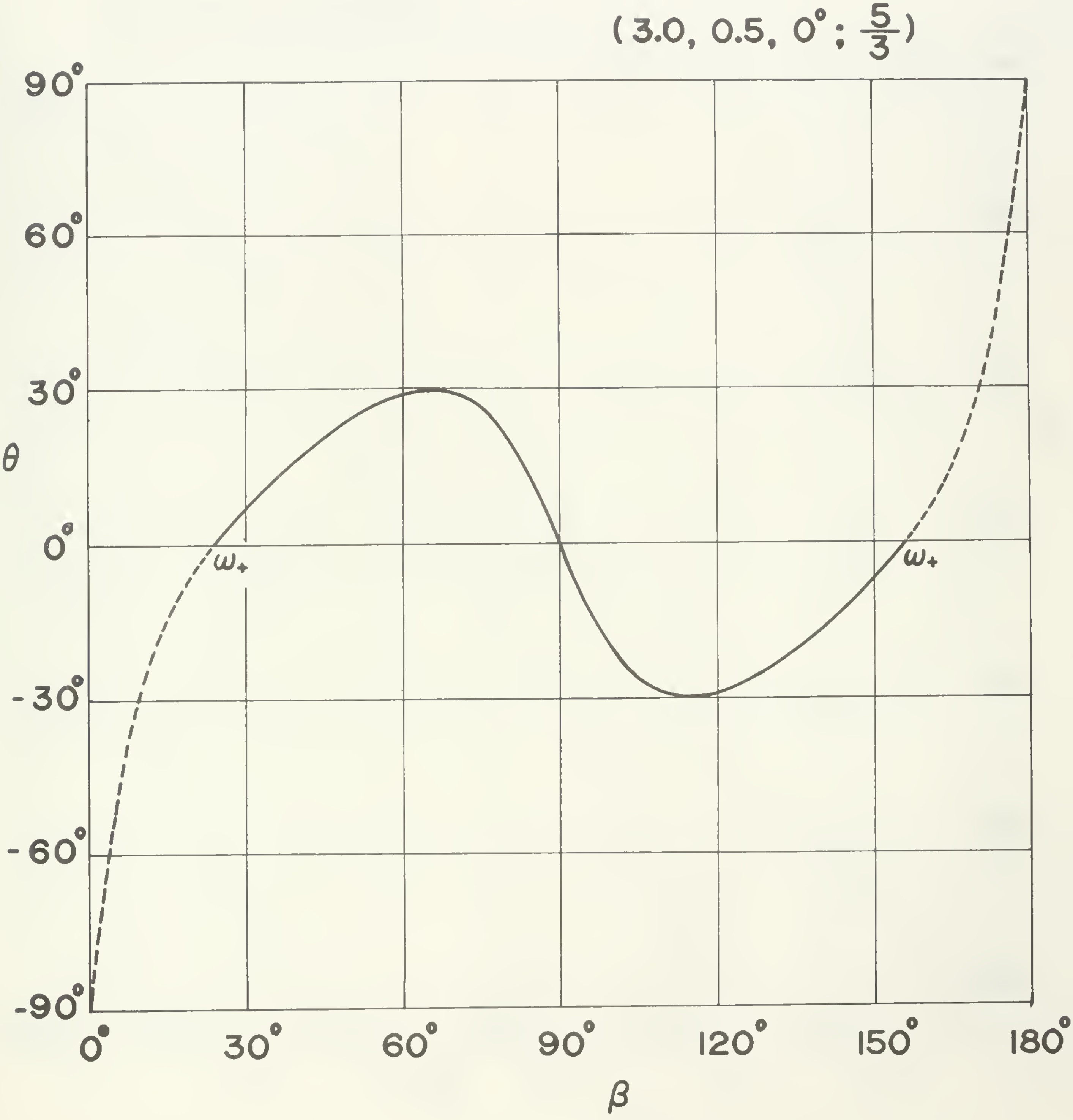
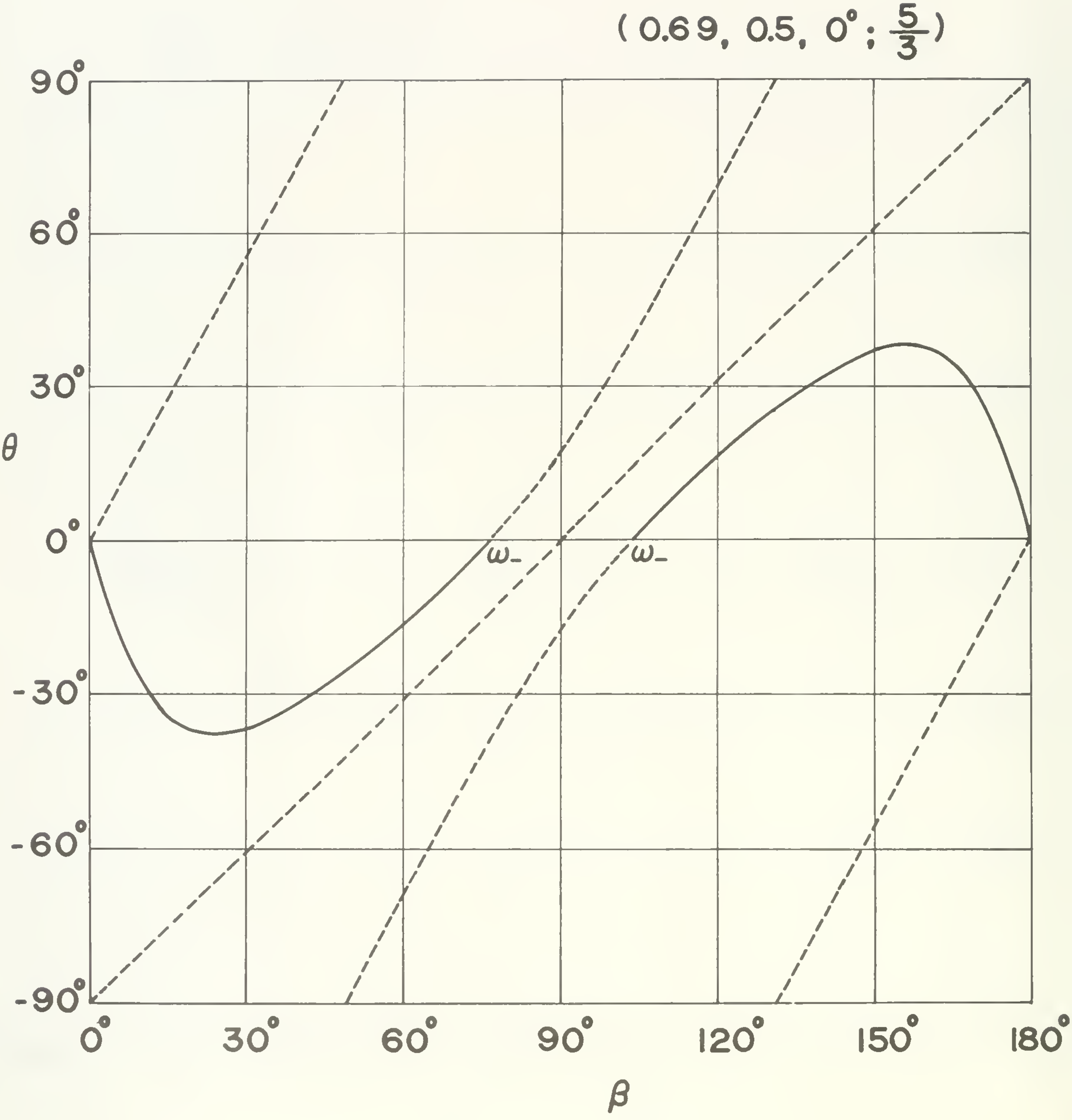


Fig. 3f


$(0.69, 0.5, 0°; \frac{5}{3})$
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60°
30°
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150°
180°
β

Fig. 3g

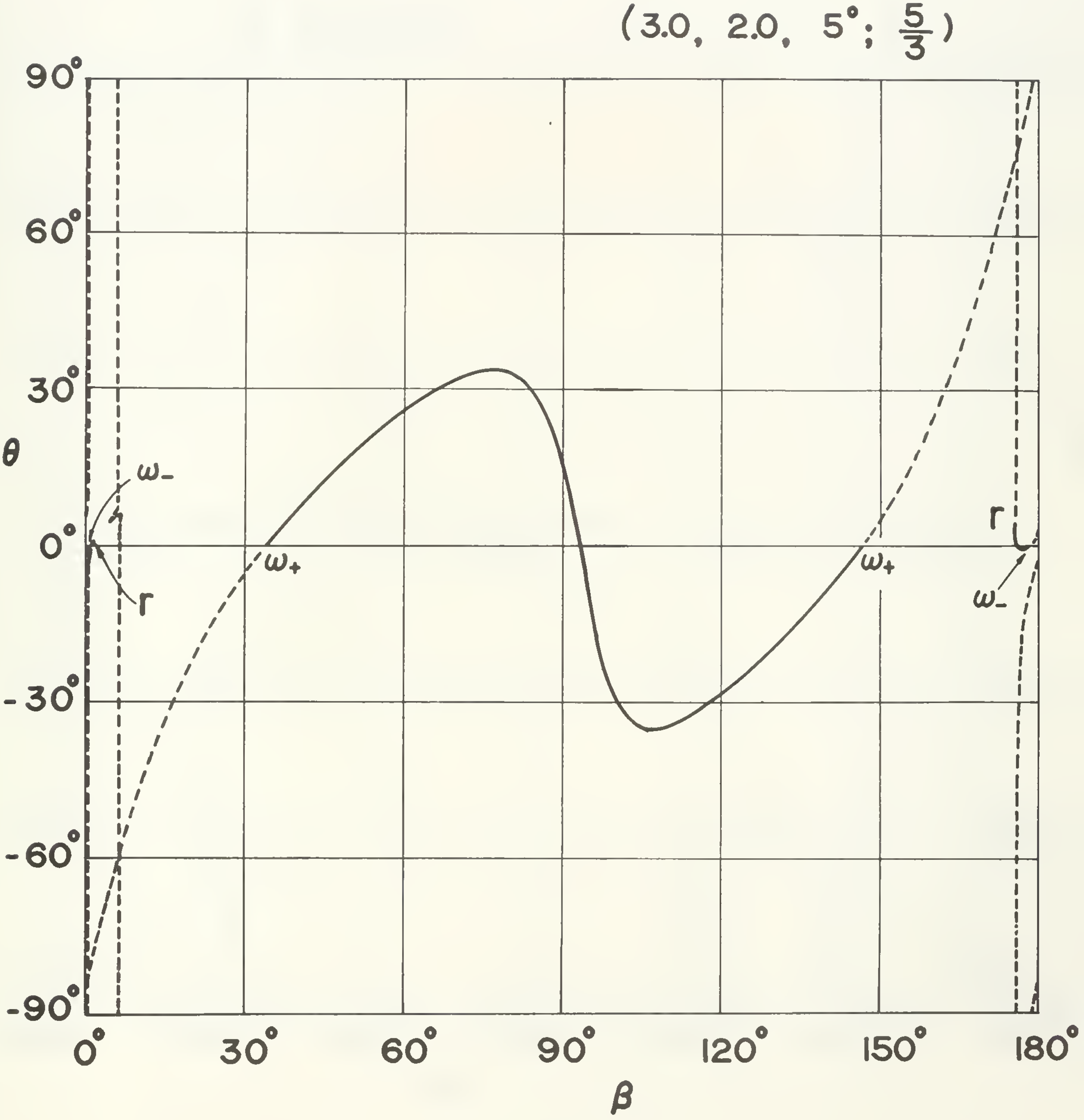

Fig. 3h

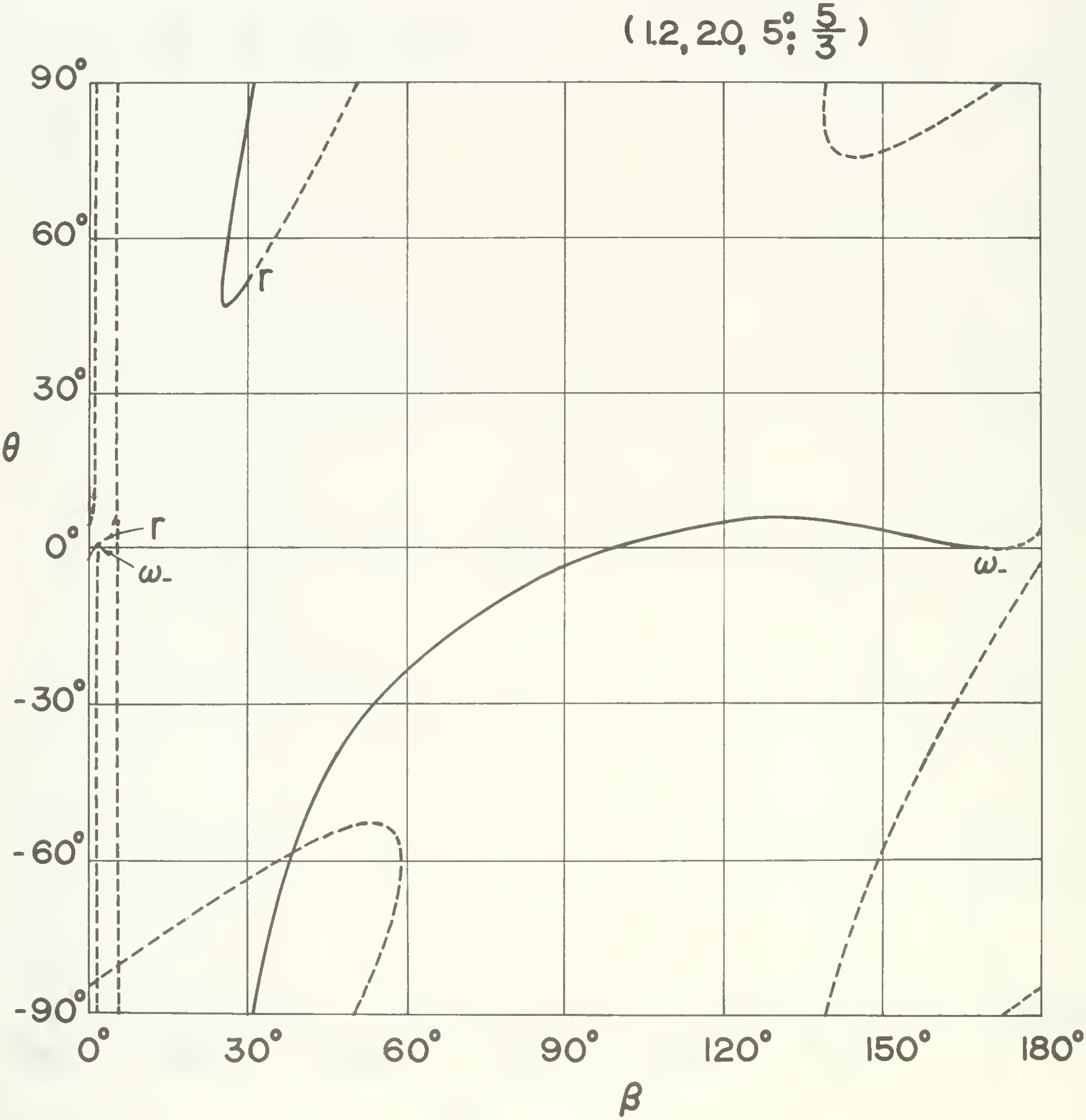

Fig. 3i

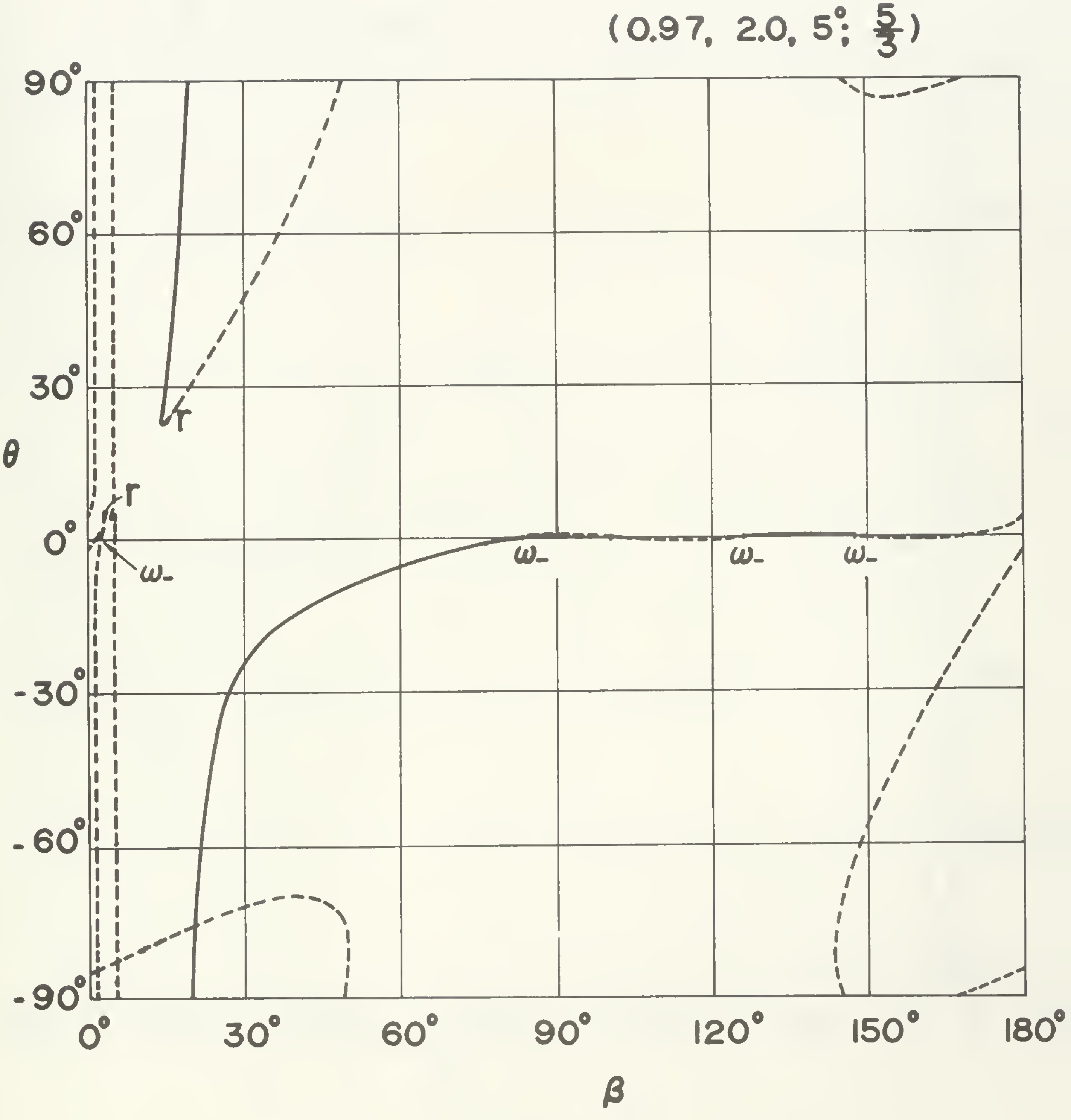

Fig. 3j

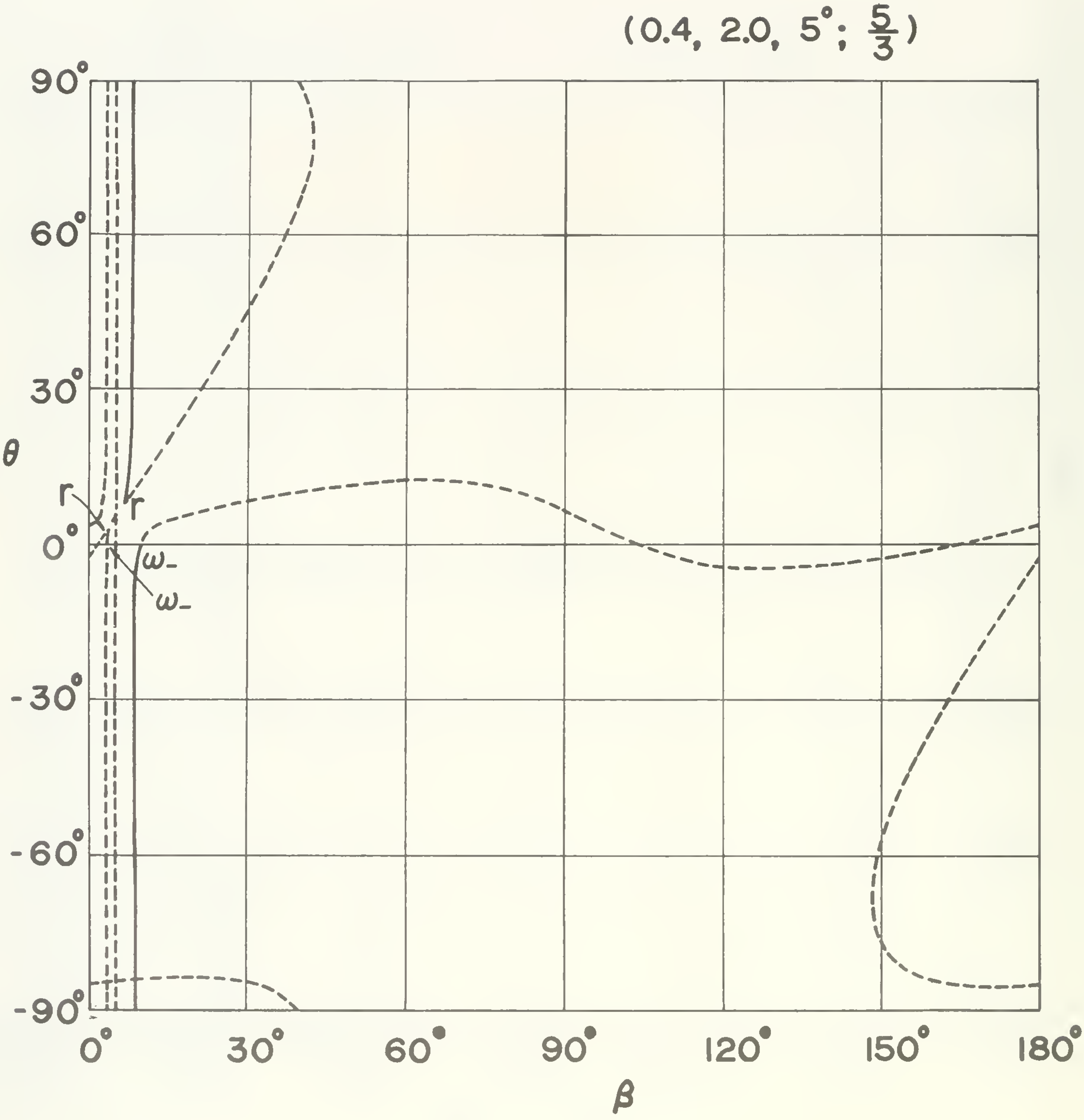

Fig. 3k

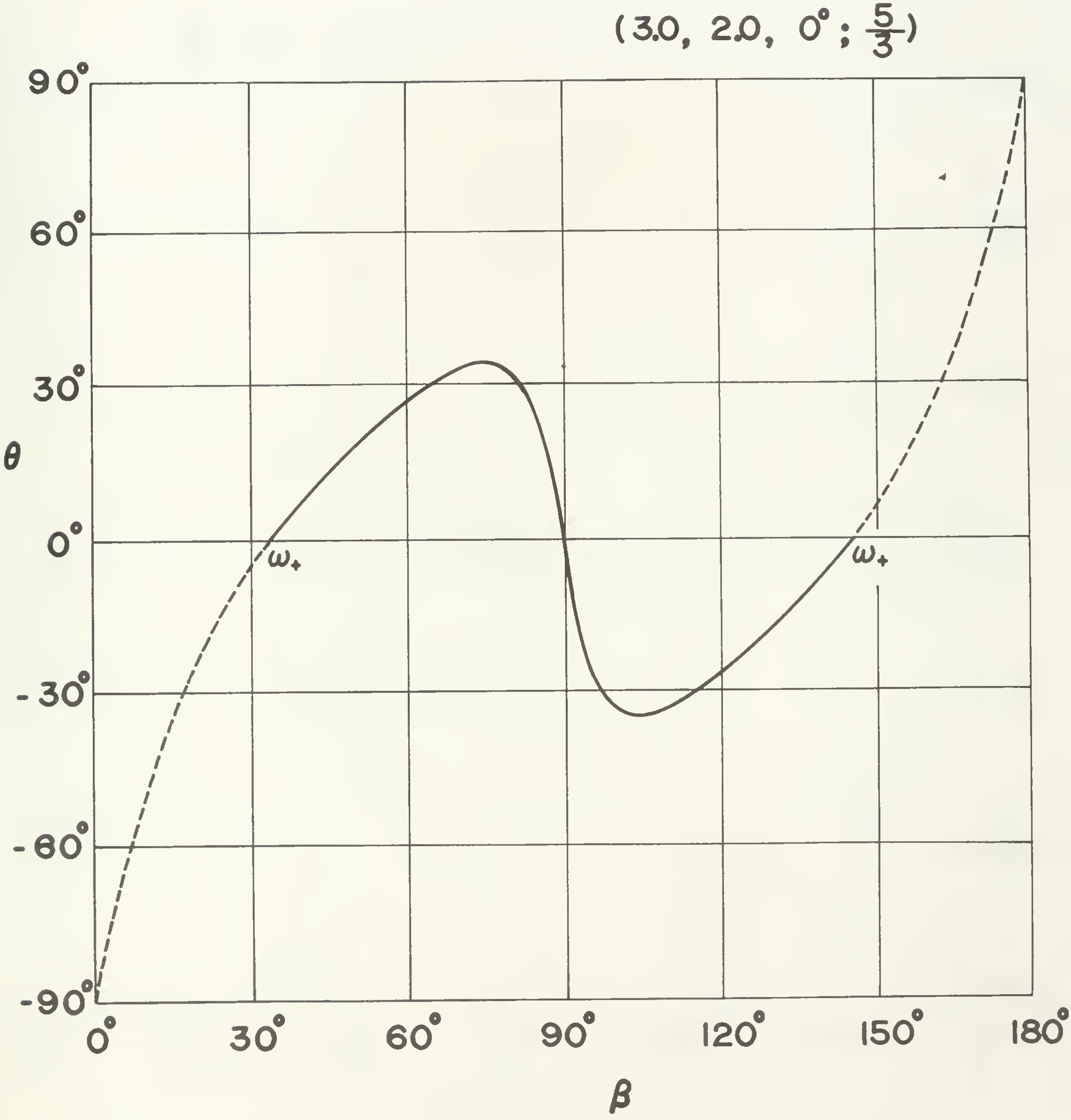

Fig. 3ℓ

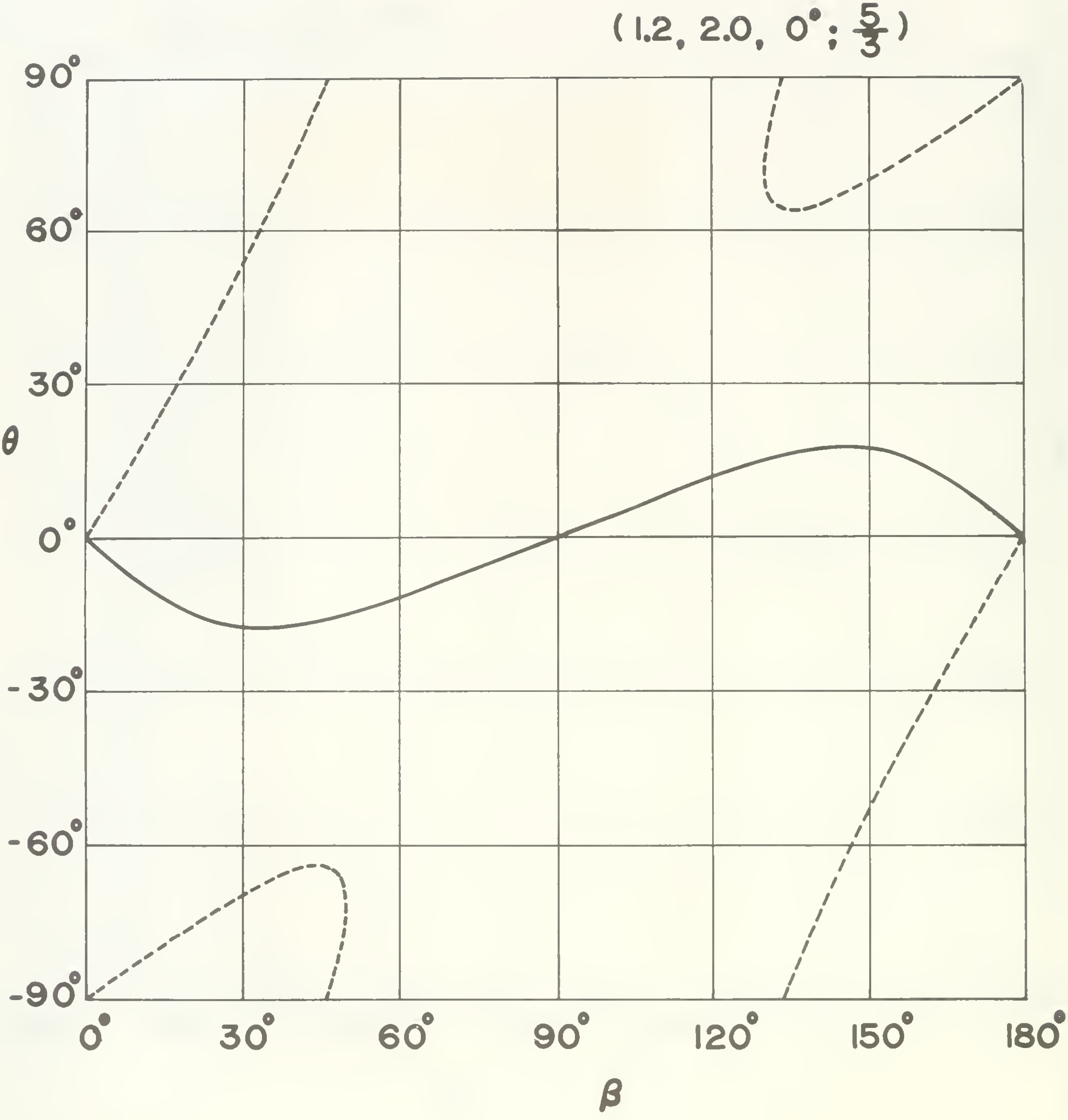

Fig. 3m

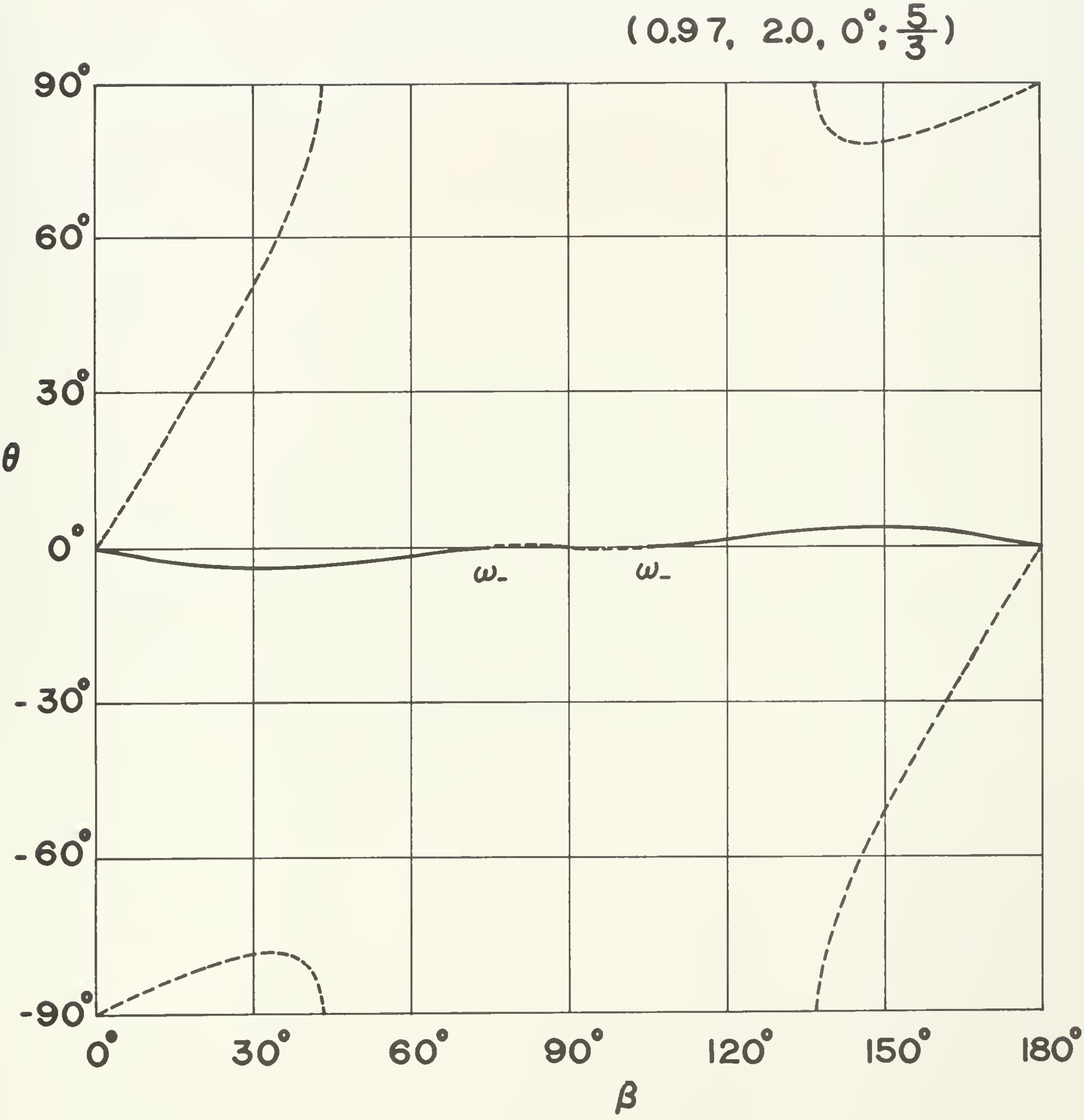

Fig. 3n

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